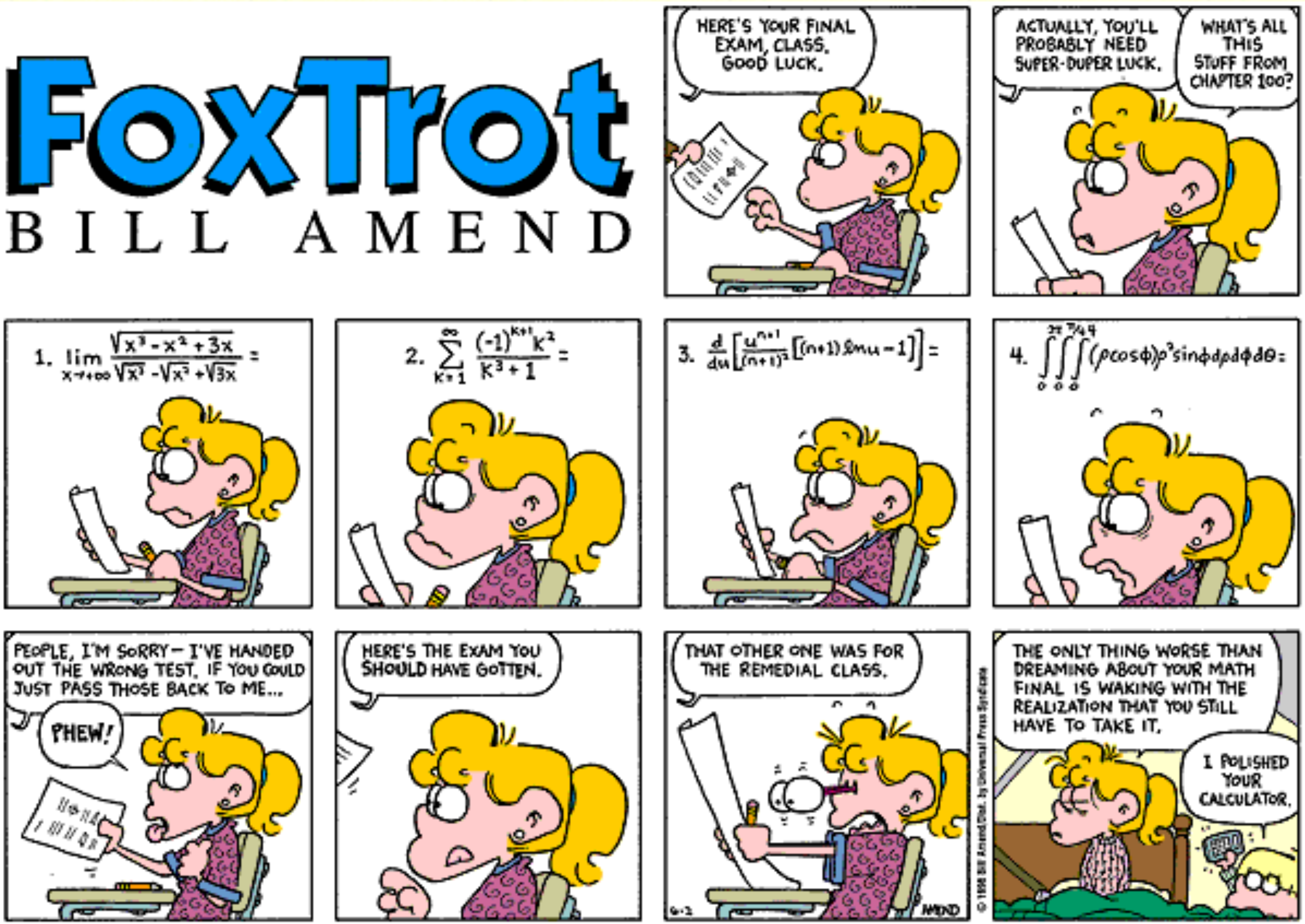

FoxTrot
BILL AMEND
HERE'S YOUR FINAL EXAM, CLASS. GOOD LUCK.
ACTUALLY, YOU'LL PROBABLY NEED SUPER-DUPER LUCK.
WHAT'S ALL THIS STUFF FROM CHAPTER 100?
1. $\lim_{x \to +\infty} \frac{\sqrt{x^3 - x^2 + 3x}}{\sqrt{x^3} - \sqrt{x^2} + \sqrt{3x}} =$
2. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2}{k^3 + 1} =$
3. $\frac{d}{du}\left[\frac{u^{n+1}}{(n+1)^2}\left[(n+1)\ln u - 1\right]\right] =$
4. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$
PEOPLE, I'M SORRY— I'VE HANDED OUT THE WRONG TEST. IF YOU COULD JUST PASS THOSE BACK TO ME...
PHEW!
HERE'S THE EXAM YOU SHOULD HAVE GOTTEN.
THAT OTHER ONE WAS FOR THE REMEDIAL CLASS.
6-2
AMEND
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THE ONLY THING WORSE THAN DREAMING ABOUT YOUR MATH FINAL IS WAKING WITH THE REALIZATION THAT YOU STILL HAVE TO TAKE IT.
I POLISHED YOUR CALCULATOR.

School of the Art Institute of Chicago

Calculus

Frank Timmes

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flash.uchicago.edu/~fxt/class_pages/class_calc.shtml

Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- www.pen.k12.va.us/Anthology/Div/Winchester/jhhs/math/humor/comics/calccom.html
- www.nku.edu/~longa/classes/2001fall/mat120/images/

Class #3

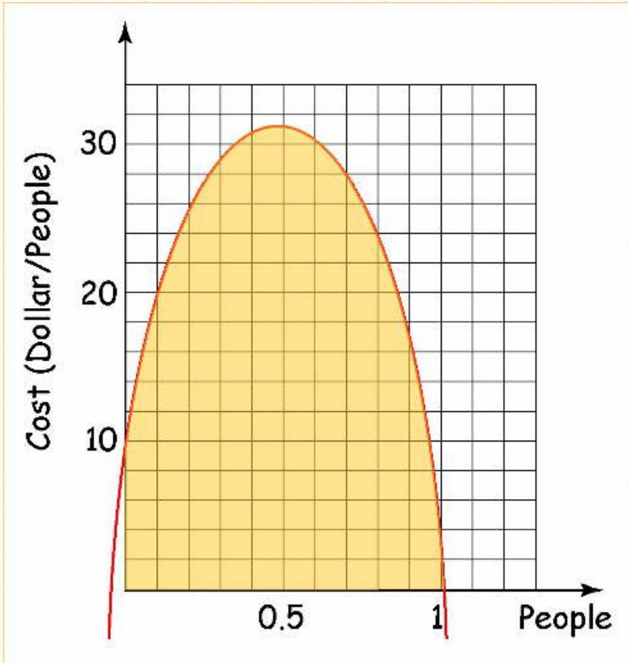
- Trapezoid rule
- Formal limits

Sum squares

- Last time we found that the definite integral of a function is the product x • y, where the y values may be different for various values of x.

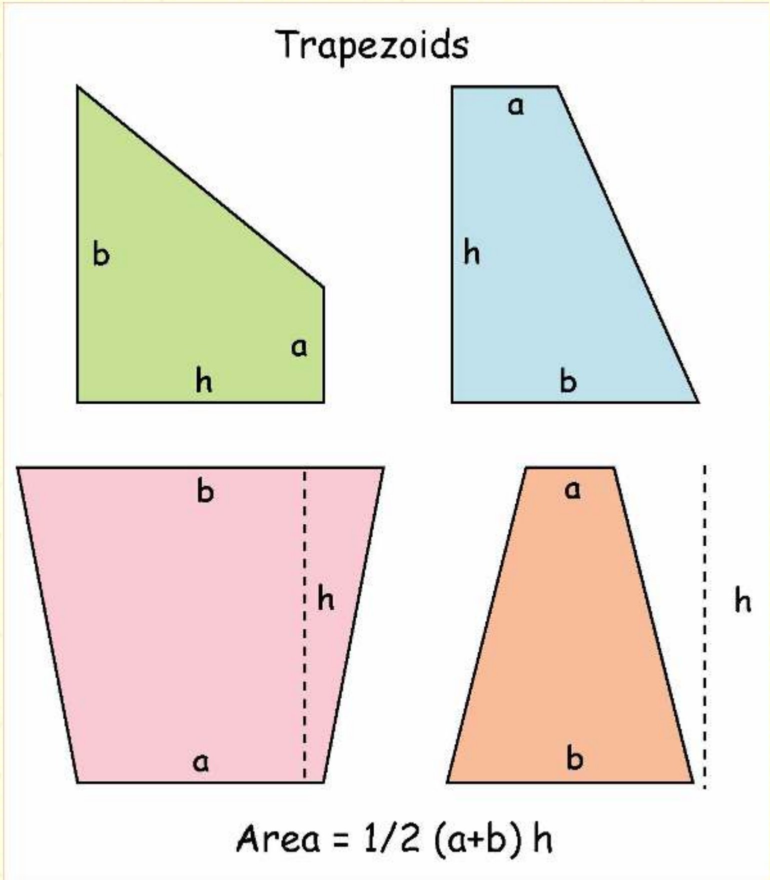
- Since the integral is represented by the area of a region under a graph, we were able to estimate it by counting squares.

- Today we'll learn a more efficient way of estimating definite integrals.



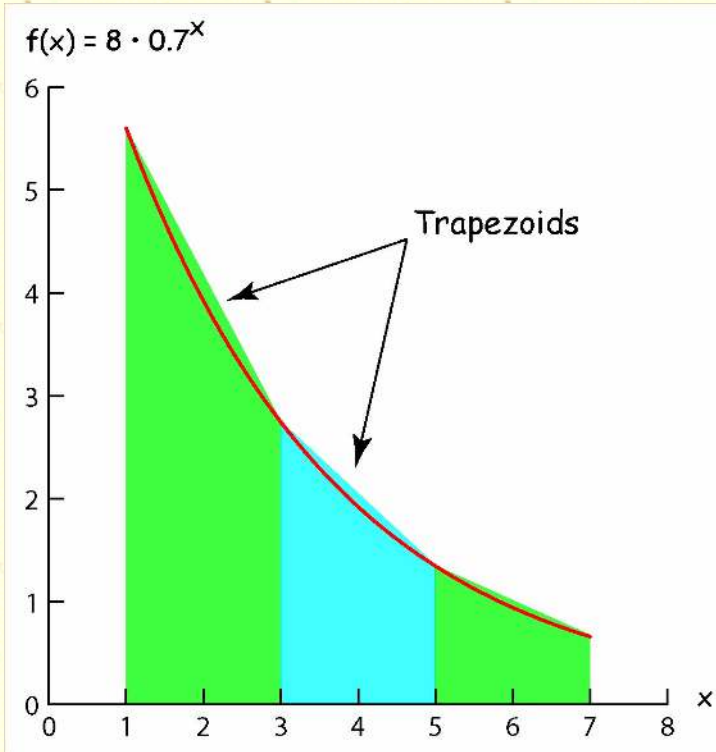
Trapezoids

- From geometry you probably recall that a trapezoid is a four-sided figure (quadrilateral) with two parallel sides.



Trapezoids

- Instead of counting squares, divide the region into vertical strips and connect the boundaries to form trapezoids.

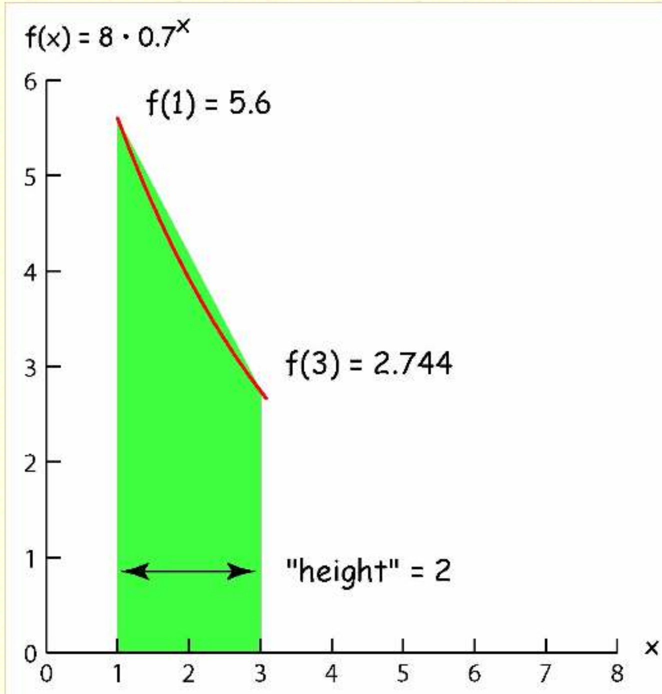


- Although the trapezoids have areas slightly different from the region under the graph, their areas are easy to calculate and add.

Trapezoids

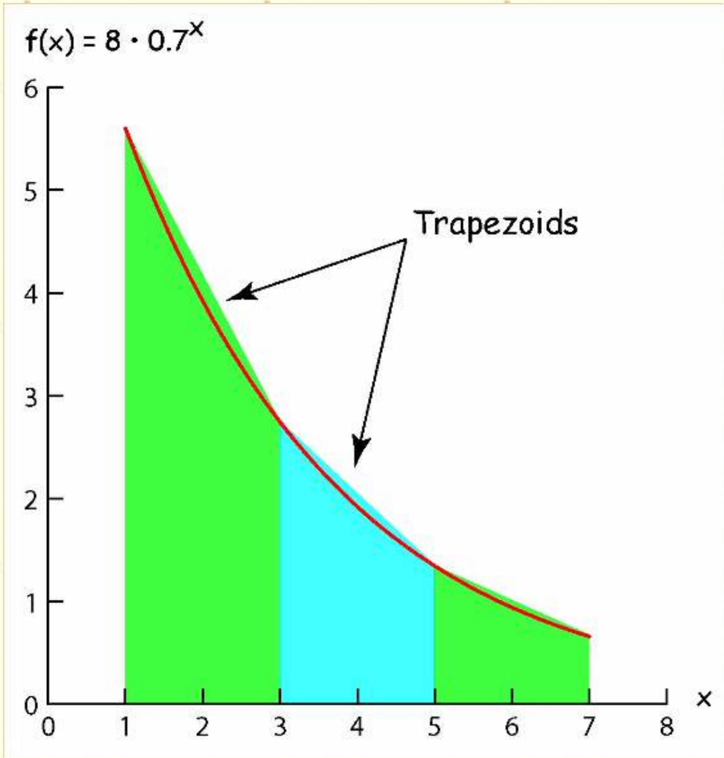
- The first trapezoid lies between x = 1 and x = 3.
  Its parallel sides are f(1) = 5.6 and f(3) = 2.744, and its height is (3 - 1) = 2.

- Thus, the area of this trapezoid is
  1/2 (5.6 + 2.744) (2) = 8.344



Trapezoids

- The areas of the other two trapezoids can be found in the same way.
- The total area of trapezoids is approximately equal to the definite integral.
- Integral
  = 8.344 + 4.08856 + 2.003394 =
  14.4359544
  = 14.4



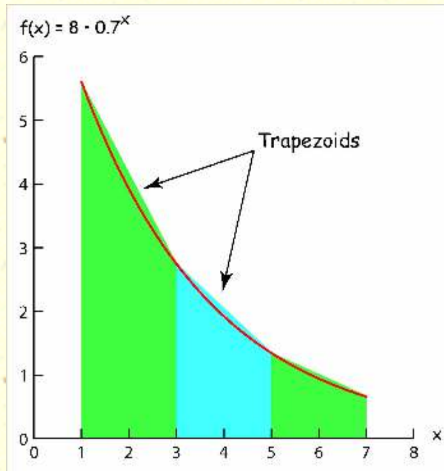
Trapezoidal rule

- To accomplish the sum in a more efficient way, observe that each y value appears twice, except for the first and last values:

$$1/2\,(5.6 + 2.744)\cdot 2 + 1/2(2.744 + 1.344)\cdot 2 + 1/2(1.344 + 0.658)\cdot 2$$

- This can be rearranged as $[1/2\cdot 5.6 + 2.744 + 1.344 + 1/2\cdot 0.658]\cdot 2$

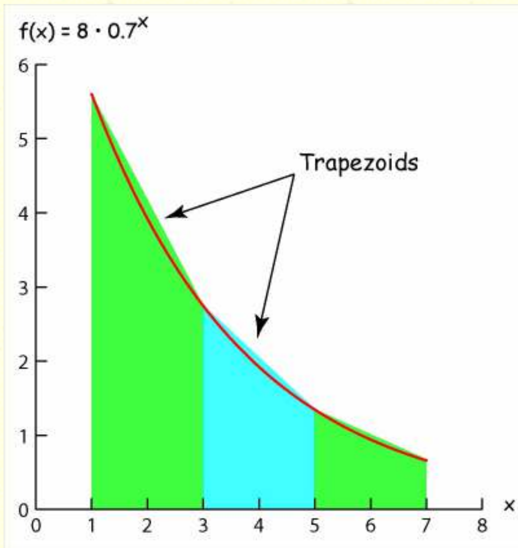
- The four terms inside the brackets are the y values at the boundaries of the four vertical strips.



Trapezoidal rule

- To find the area you add the y values, taking half the first and half the last one, and multiply by the width of each strip.

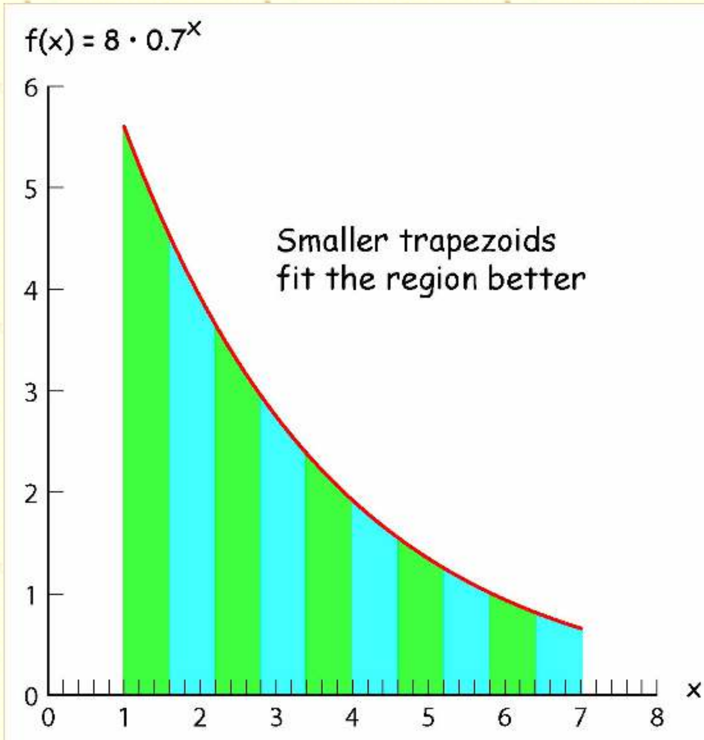
$$\text{Integral} \approx \left[\frac{1}{2} f(x_1) + f(x_2) + f(x_3) + \ldots + \frac{1}{2} f(x_N)\right] \cdot \text{width}$$



Trapezoidal rule

- This procedure for finding the approximate value of a definite integral is called the trapezoidal rule.

- Let's do the same problem, but using 10 increments.



Trapezoidal rule

x	f(x)	
1.0	5.6	Use half of this one
1.6	4.5211	
2.2	3.6501	
2.8	2.9468	
3.4	2.3791	
4.0	1.9208	
4.6	1.5507	
5.2	1.2519	
5.8	1.0107	
6.4	0.8160	
7.0	0.6588	Use half of this one

- Without rounding, add the y values as you calculate them: sum = 23.1770
- Integral = sum · width
  = 23.1770 · 0.6
  = 13.9062452

Trapezoidal limit

- Our new answer of 13.9062 is smaller than our previous answer of 14.4 because the smaller trapezoids fit the region under the graph better.

- If we were to increase the number of strips, the value for the integral would get closer and closer to 13.8534. This number is the limit of the areas of the trapezoids as their widths approach zero (infinite number of trapezoids).

- Later in this course, we'll learn to calculate exact values of various integrals.

Limit of a function

- The two crown jewels of calculus, derivatives and integrals, hinge upon the concept of the limit of a function.

- Last time we introduced a verbal definition of a limit:

  L is the limit of f(x) as x approaches c
  if and only if
  L is the one number you can keep f(x) arbitrarily close to,
  just by keeping x close enough to c, but not equal to c.

Limit of a function

- Over the next couple of classes we'll learn a formal definition, and see how this is related to the physical meaning of a limit.

- For now, given a graph or equation of a function, our objective is tell whether or not the function has a limit as x approaches the given value and tell how our answers relate to the definition of a limit.

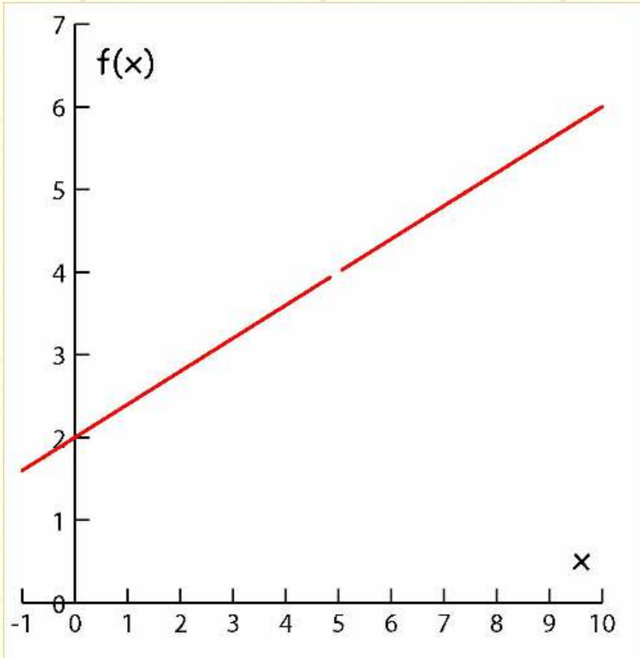
Circle Limit IV
1959, M.C. Escher

Limit of a function

- Consider the rational algebraic function

$$f(x) = \frac{0.4\,x^2 - 10}{x - 5}$$

- Its graph is a straight line with a gap.

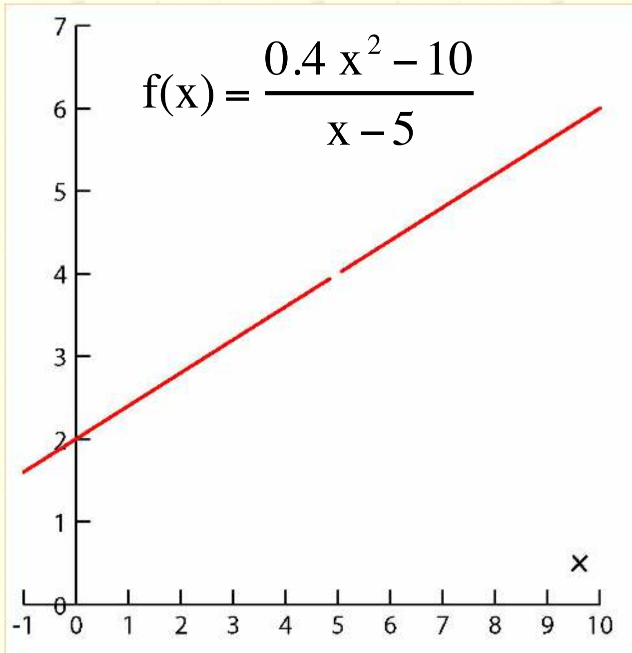


- The gap occurs at x = 5, and comes from the denominator being zero when x = 5.

Limit of a function

- Using an x increment of 0.1, we find the values shown. No value appears where x = 5, but the pattern suggests that it should be exactly 4 when x = 5.

x	f(x)
4.6	3.84
4.7	3.88
4.8	3.92
4.9	3.96
5.0	
5.1	4.04
5.2	4.08



Limit of a function

- Try to find f(5) by direct substitution:

$$f(5) = \frac{0.4 \cdot 5^2 - 10}{5 - 5} = \frac{0}{0}$$

which is undefined because of the division by zero.

$$f(x) = \frac{0.4\,x^2 - 10}{x - 5}$$

- Algebra shows us what is happening:

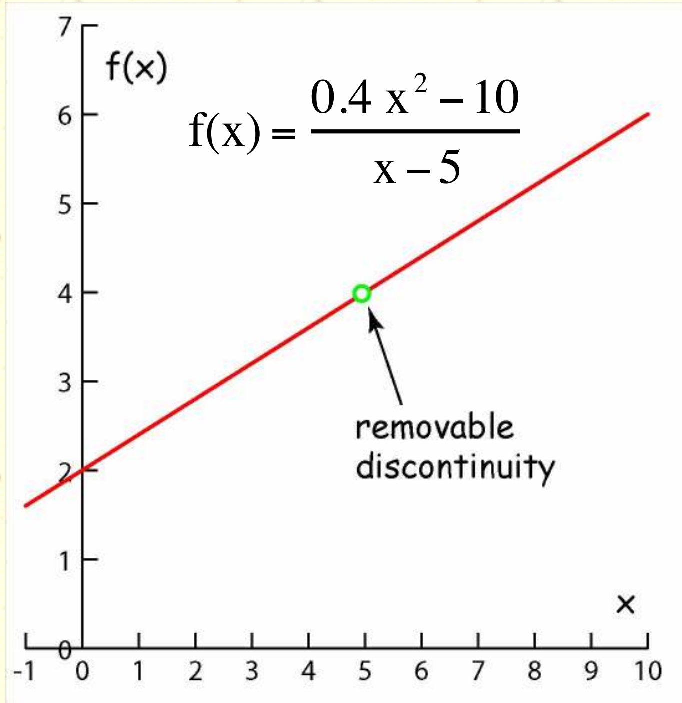
$$f(x) = \frac{0.4\,(x + 2)(x - 5)}{x - 5}$$

$$f(x) = 0.4x + 2$$

- Now f(5) = 0.4•5 + 2 = 4, as the pattern in our previous table suggested.

Limit of a function

- The function is said to have a removable singularity at x = 5.
- The function is discontinuous because of the gap, but the gap can be removed by defining f(5) = 4.
- Traditionally one draws an open circle at a discontinuity, indicating there is no value of y for that value of x.



Limit of a function

L is the limit of f(x) as x approaches c
if and only if
L is the one number you can keep f(x) arbitrarily close to,
just by keeping x close enough to c, but not equal to c.

- The number 4, which f(x) is close to when x is close to 5, is the limit of f(x) as x approaches 5. We can make f(x) as close as you like to 4 just by keeping x close enough to 5 (but not equal to 5).

- Can you see the pieces of the verbal definition of a limit coming together?

Limit of a function

- Suppose someone tells you "Keep f(x) within 0.01 units of 4".

x	f(x)
4.970	3.988
4.975	3.990
4.980	3.992
4.985	3.994
4.990	3.996
4.995	3.998
5.0	
5.005	4.002
5.010	4.004
5.015	4.006
5.020	4.008
5.025	4.010
5.030	4.012

- To find approximately how close you must keep x to 5, you can make a graph or table.

x within 0.025 units of 5

y within 0.01 units of 4

Limit of a function

- We can also use algebra to find out exactly how close to keep x to 5.

$$3.99 < f(x) < 4.01$$

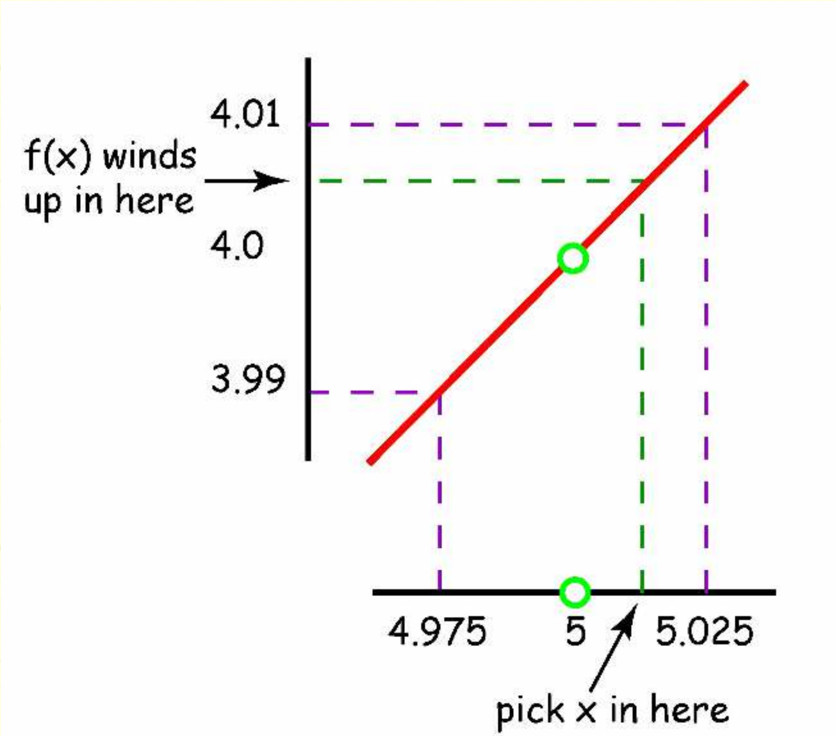
$$3.99 < 0.4x + 2 < 4.01$$

$$1.99 < 0.4x < 2.01$$

$$4.975 < x < 5.025$$

- The last inequality says that x must be within 0.025 units of 5.
  By reversing the steps you can see that if x is within 0.025 units of 5, then f(x) is within 0.01 units of 4.

Limit of a function



- These investigations lead to the following formal definition of a limit.

Formal definition

L is the limit of f(x) as x approaches c
if and only if
for any positive number epsilon ε, no matter how small,
there is a positive number delta δ such that
if x is within δ units of c (but not equal to c),
then f(x) is within ε units of L.

- You should be able to make the following connections with our example:
  - 4 is the value of L.
    It's the one number f(x) stays arbitrarily close to if x is close to 5.
  - 5 is the value of c.
    It's the number you keep x close to in order for f(x) to stay close to 4.

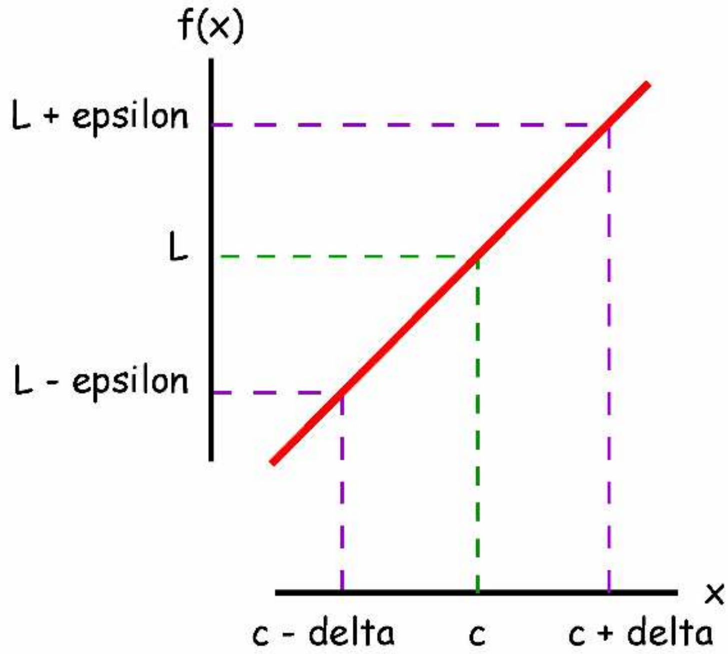
Formal definition

L is the limit of f(x) as x approaches c
if and only if
for any positive number epsilon ε, no matter how small,
there is a positive number delta δ such that
if x is within δ units of c (but not equal to c),
then f(x) is within ε units of L.

- 0.01 is the value of epsilon ε.
  It is picked arbitrarily to tell how close to 4 you are supposed to keep f(x).

- 0.025 is the corresponding value of delta δ.
  It tells how close to 5 is "close enough" to keep x,
  in order that f(x) winds up somewhere within 0.01 units of 4.

Formal definition

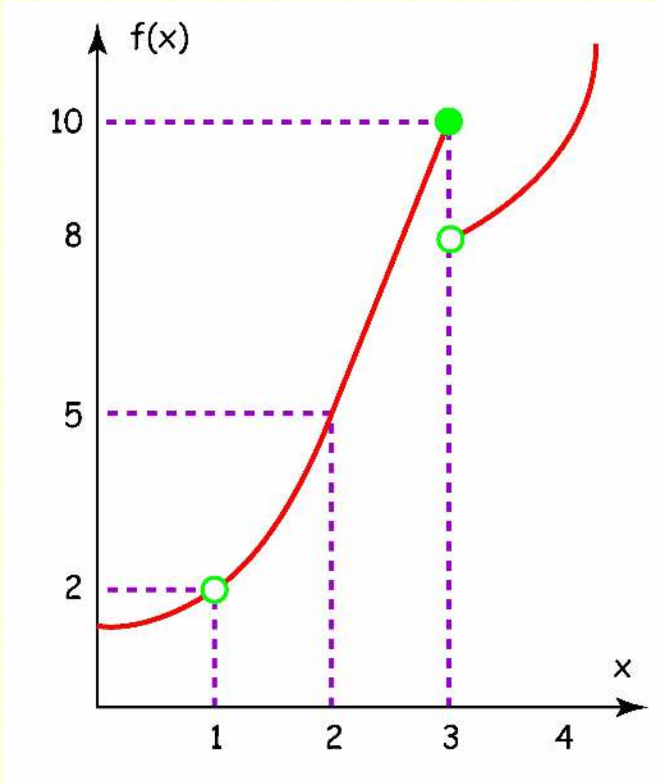


Given f(x), c, and epsilon,
find delta, if it exists

Limits

- Let's do an example.
  Is there a limit at x = 1?
  Why or why not?

- As x approaches 1, the limit is 2. If x is close to 1, but not equal to 1, f(x) is close to 2.

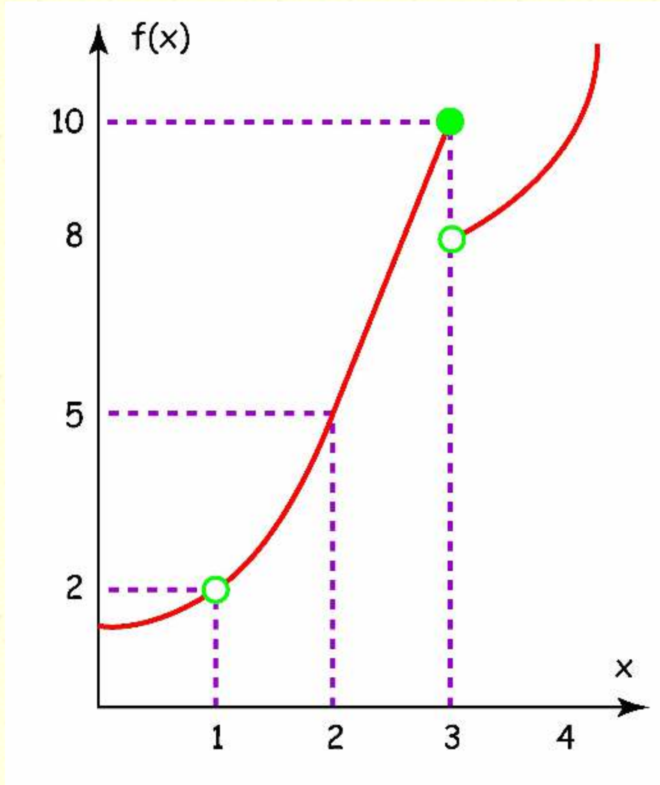


Limits

- Is there a limit at x=2?
  Why or why not?

- As x approaches 2, the limit is 5. If x is close to 2, but not equal to 2, f(x) is close to 5.

- The fact that x can equal 2 is of no consequence!



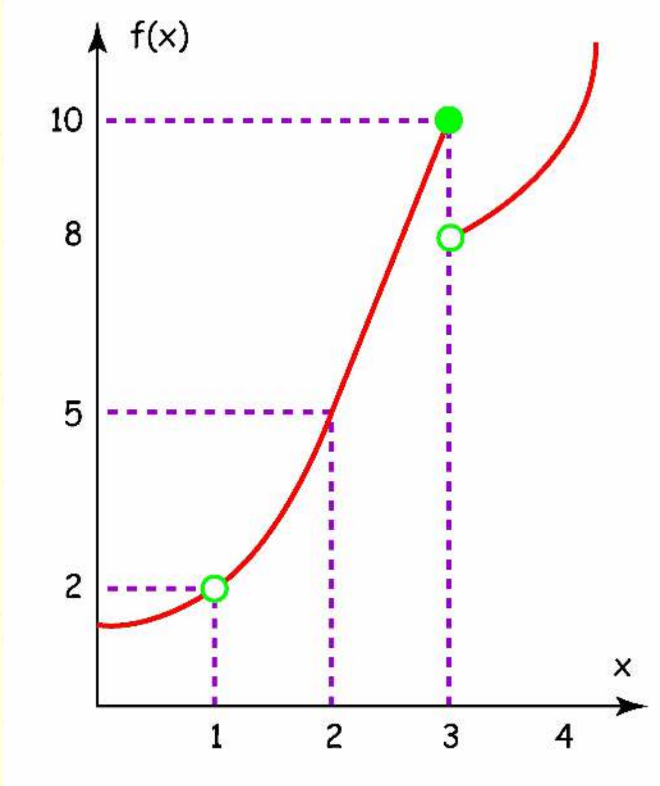
Limits

- Is there a limit at x=3?
  Why or why not?

- As x approaches 3, there is no limit. If x is close to 3 on the left, f(x) is close to 10. If x is close to 3 on the right, f(x) is close to 8.

- So there isn't one number f(x) that can be kept close to by keeping x close to 3 but not equal to 3.

- This type of discontinuity is called a step discontinuity.



Playtime

- During your in-class problem solving session today you'll estimate integrals by the trapezoidal rule and investigate some limits of functions.