The longer mathematics lives, the more abstract – and therefore, possibly also the more practical – it becomes.

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Calculus

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Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- www.karlscalculus.org/calculus.html

- archives.math.utk.edu/visual.calculus/1/definition.6/

Class #4

- Limits II
- Continuity

Limits

- We introduced a verbal definition of a limit:

> L is the limit of f(x) as x approaches c
> if and only if
> L is the one number you can keep f(x) arbitrarily close to,
> just by keeping x close enough to c, but not equal to c.

- Last time we formalized this a bit:

> L is the limit of f(x) as x approaches c
> if and only if
> for any positive number epsilon ε, no matter how small,
> there is a positive number delta δ such that
> if x is within δ units of c (but not equal to c),
> then f(x) is within ε units of L.

Limits

- We can shorten the definition more by using algebraic symbols and leaving out words that were added for clarity.

- The word limit is abbreviated lim, and the words as x approaches c are expressed as x -> c.

Limits

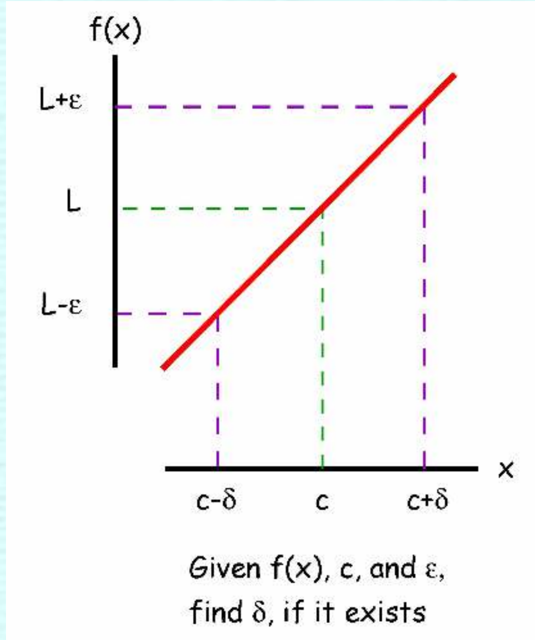
$$L = \lim_{x \to c} f(x) \text{ if and only if}$$

for any number $\varepsilon > 0$,

there is a number $\delta > 0$ such that

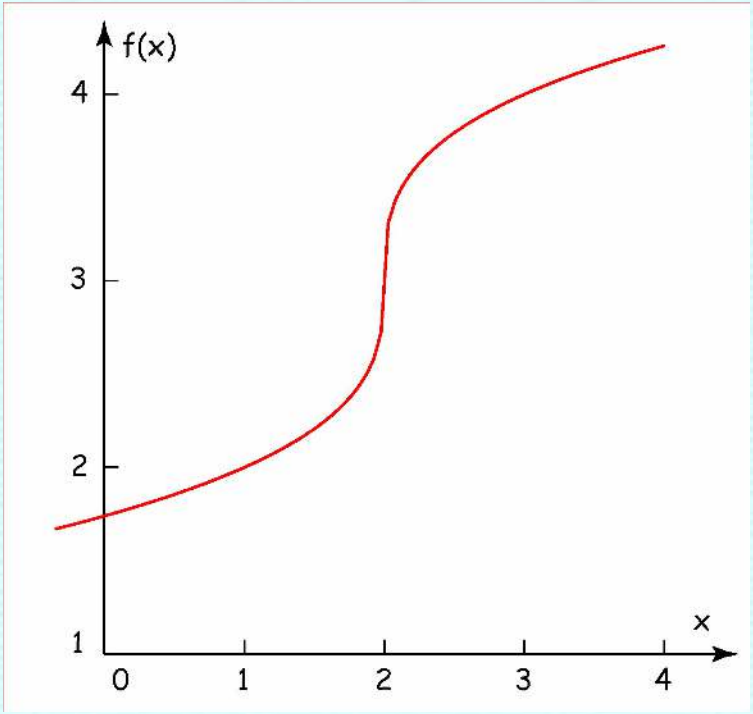
if x is within δ units of c (but $x \neq c$),

then f(x) is within ε units of L.



Example

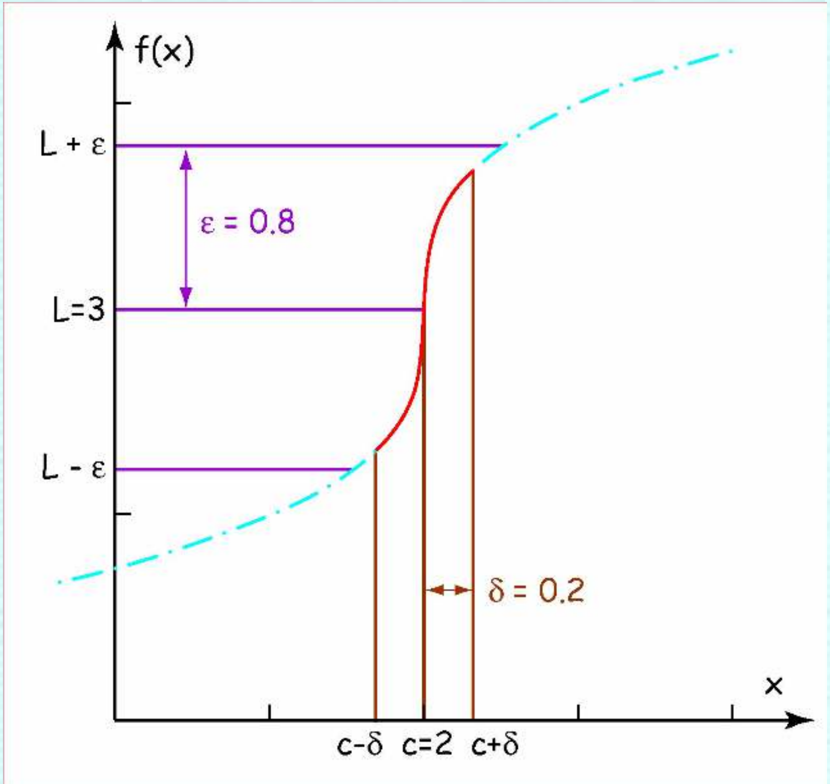
- Consider $f(x) = (x - 2)^{1/3} + 3$



- Let's look in the vicinity of $c = 2$, and $f(x=c) = 3$

Too small

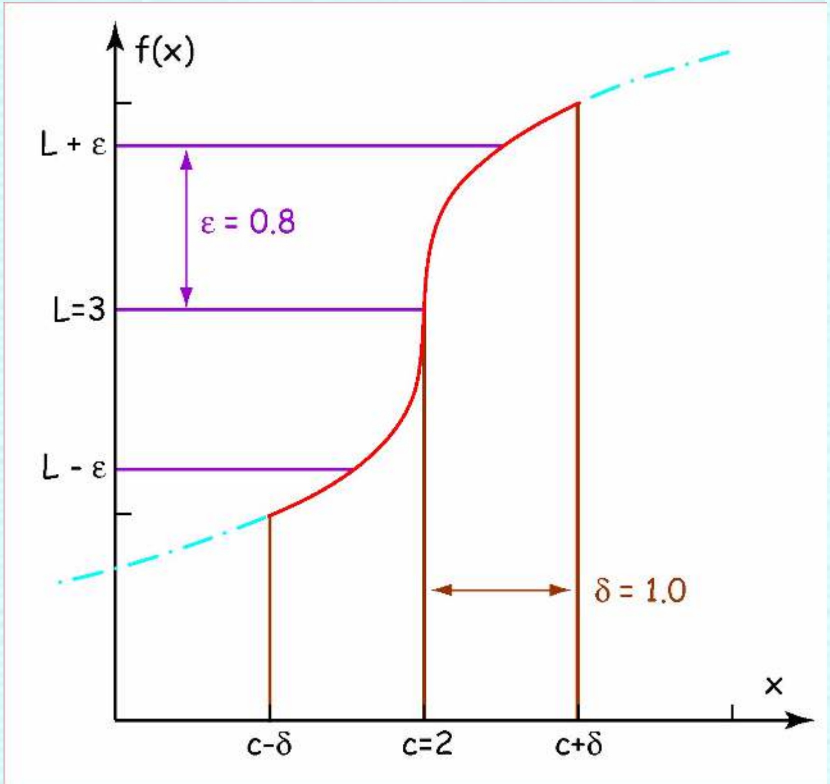
- To see how to apply the limit definition, let ε = 0.8
- The figure shows what happens if we pick δ = 0.2
- All the values of f(x) are between the horizontal lines at y = 2.2 and 3.8.
- delta δ is small enough.
  The whole graph is between the horizontal lines.



Too large

- If $\delta = 1.0$ some of the corresponding values of f(x) will be above or below the horizontal lines.

- Thus, 1.0 is too big to be a suitable value of δ in this case.



Just right

- To figure out a value of δ that is "just right", one can use a graphical technique to get an approximate value, or algebra to find the exact answer.

- We want the function $f(x) = (x-2)^{1/3} + 3$ to be between $3 - \varepsilon$ and $3 + \varepsilon$. Write an equality and solve for x:

$$3-\varepsilon < (x-2)^{1/3}+3 < 3+\varepsilon$$

$$-\varepsilon < (x-2)^{1/3} < \varepsilon$$

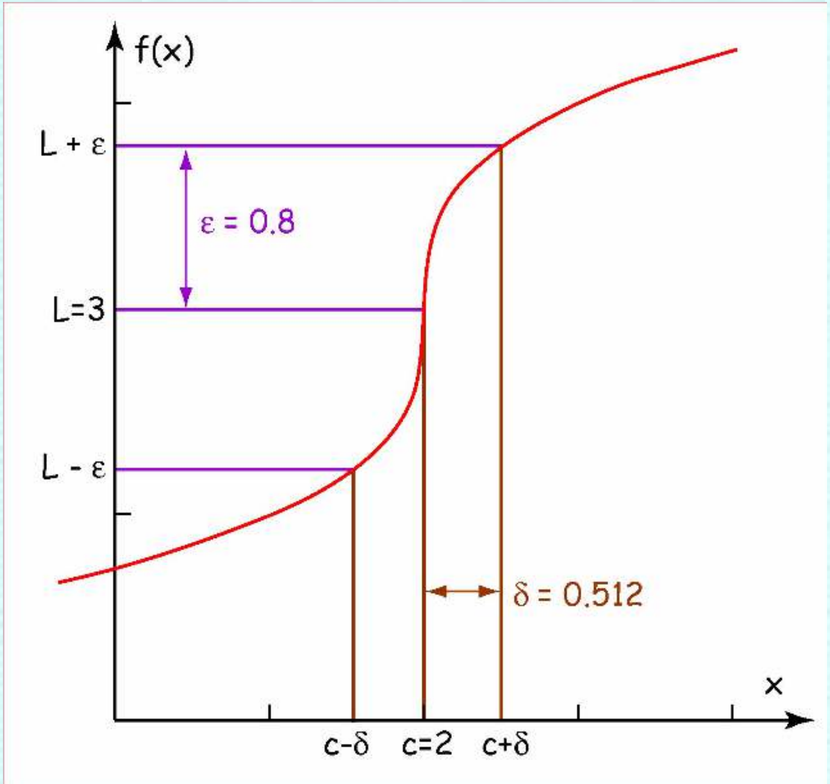
$$-\varepsilon^3 < x-2 < \varepsilon^3$$

$$2-\varepsilon^3 < x < 2+\varepsilon^3$$

- Thus, the largest possible value of δ is ε^3.

Just right

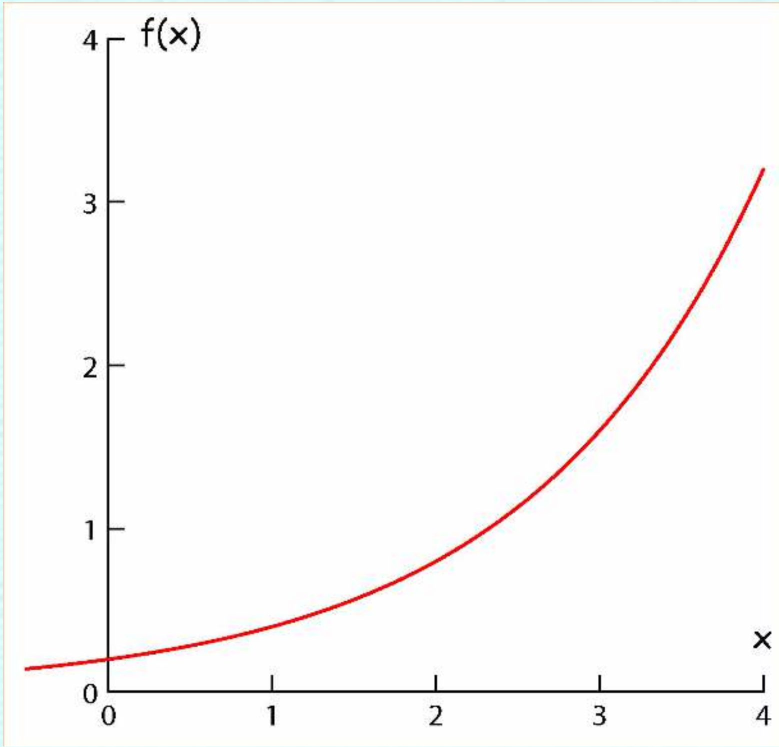
- For $\varepsilon = 0.8$ the value $\delta = 0.512$ is just right.



Example

- Let $f(x) = 0.2 \cdot 2^x$

1. What is the limit of $f(x)$ as x approaches 3?

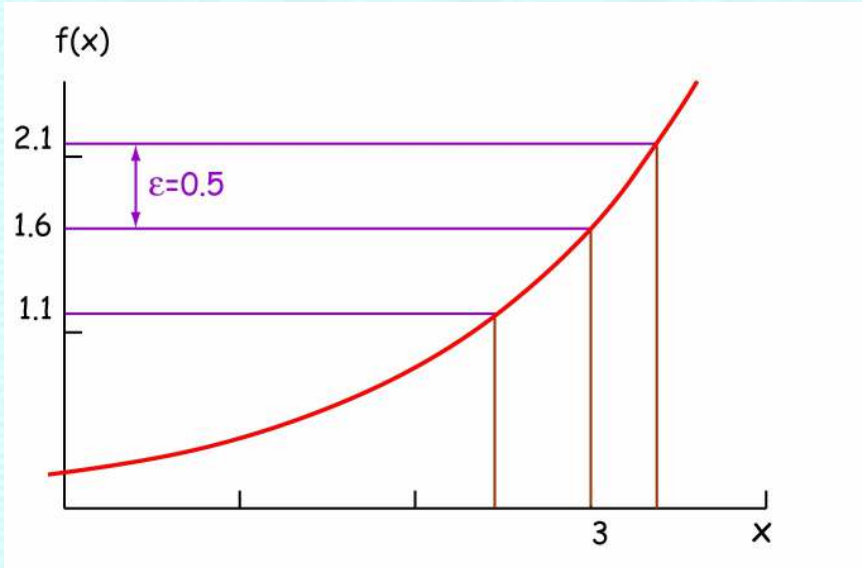


Because the plot has no discontinuities, the limit is 1.6, the same as $f(3)$.

Find delta

2. Find the maximum value of δ that can be used for $\varepsilon = 0.5$ at $x = 3$.

The horizontal lines at y = 1.1 and y = 2.1 show that we want to keep f(x) within 0.5 units of 1.6.



To find the largest possible value of δ, we have to find the values of x where the horizontal lines cross the graph.

Find delta

For the case when $f(x) = L - \varepsilon = 1.6 - 0.5 = 1.1$, algebra gives

$$0.2\,(2^{x}) = 1.1$$

$$2^{x} = 5.5$$

$$\log 2^{x} = \log 5.5$$

$$x\ \log 2 = \log 5.5$$

$$x = \frac{\log 5.5}{\log 2} = 2.4594\ldots$$

So δ could be $3 - 2.4594 = 0.54$

Find delta

Similarily, when $f(x) = L + \varepsilon = 1.6 + 0.5 = 2.1$, algebra yields

$$0.2\,(2^x) = 2.1$$

$$2^x = 10.5$$

$$\log 2^x = \log 10.5$$

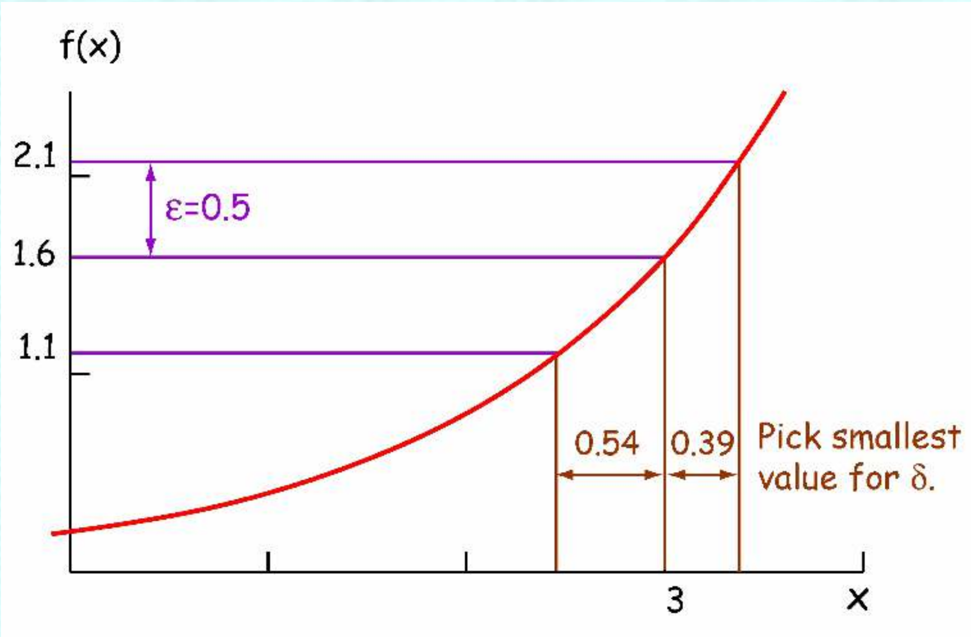
$$x \log 2 = \log 10.5$$

$$x = \frac{\log 10.5}{\log 2} = 3.3923\ldots$$

So δ could be 3.3923 - 3 = 0.39

Find delta

The one value of δ in the definition of limit will be the smaller of these two, $\delta = 0.39$



Find delta

3. Show how to find a positive δ for any ε > 0.

We just repeat the calculation done before, but using ε instead of 0.5.

Picking δ on the steeper side of $x = 3$ means $f(x)$ will equal $1.6 + \varepsilon$.

$$0.2\,(2^{x}) = 1.6 + \varepsilon$$

$$2^{x} = 8 + 5\varepsilon$$

$$\log 2^{x} = \log(8 + 5\varepsilon)$$

$$x \log 2 = \log(8 + 5\varepsilon)$$

$$x = \frac{\log(8 + 5\varepsilon)}{\log 2}$$

$$\delta = \frac{\log(8 + 5\varepsilon)}{\log 2} - 3$$

Limit theorems

- Limit of the identity function: $$\lim_{x \to c} x = c$$

Informal: The limit of x as x approaches c is simply c.

- Limit of a constant function: $$\text{if } (x) = k, \text{ where k is a constant,}$$

$$\text{then } \lim_{x \to c} f(x) = k$$

Informal: The limit of a constant is that constant.

Limit theorems

- Limit of a constant times a function:

$$\text{If } \lim_{x \to c} g(x) = L$$

$$\text{then } \lim_{x \to c} [k \cdot g(x)] = k \cdot \lim_{x \to c} g(x) = k \cdot L$$

Informal: The limit of a constant times a function equals the constant times the limit.

Limit theorems

- Limit of a sum of two functions:

$$\text{If } \lim_{x \to c} g(x) = L_1 \text{ and } \lim_{x \to c} h(x) = L_2,$$

$$\text{then } \lim_{x \to c} [g(x) + h(x)] = \lim_{x \to c} g(x) + \lim_{x \to c} h(x) = L_1 + L_2$$

Informal: Limits distribute over addition;
the limit of a sum equals the sum of the limits.

Limit theorems

- Limit of a product of two functions:

$$\text{If } \lim_{x \to c} g(x) = L_1 \text{ and } \lim_{x \to c} h(x) = L_2,$$

$$\text{then } \lim_{x \to c} [g(x) \cdot h(x)] = \lim_{x \to c} g(x) \cdot \lim_{x \to c} h(x) = L_1 \cdot L_2$$

Informal: Limits distribute over multiplication;
the limit of a product equals the product of the limits.

Limit theorems

- Limit of a quotient of two functions:

$$\text{If } \lim_{x\to c} g(x) = L_1 \text{ and } \lim_{x\to c} h(x) = L_2, \text{where } L_2 \neq 0,$$

$$\text{then } \lim_{x\to c} \frac{g(x)}{h(x)} = \frac{\lim_{x\to c} g(x)}{\lim_{x\to c} h(x)} = \frac{L_1}{L_2}$$

Informal: Limits distribute over division, except for division by zero; the limit of a quotient equals the quotient of the limits.

Limit theorems

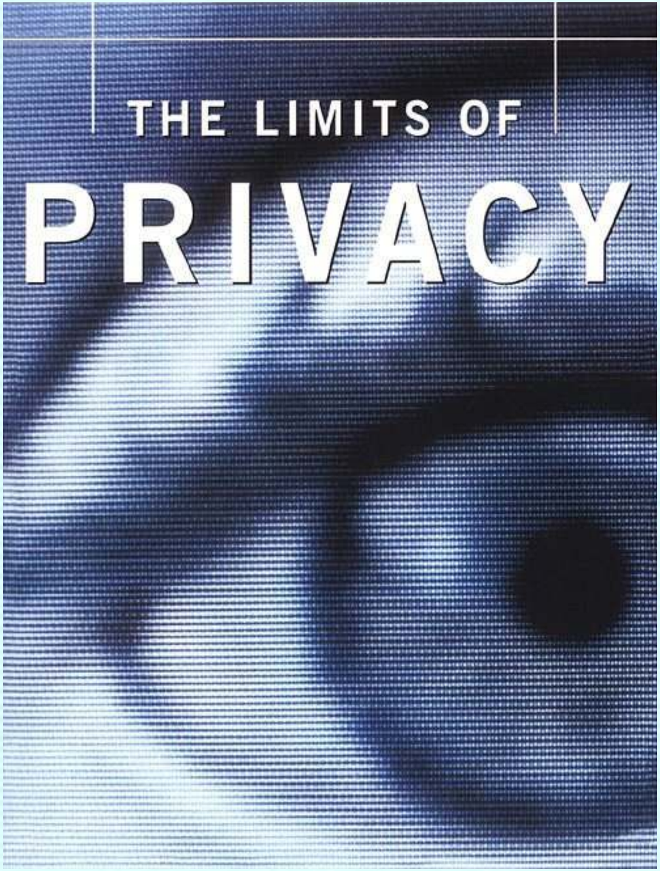
- Limit of a composite function:

$$\text{if } x \to c \;\Rightarrow\; u \to k,$$
$$\text{then } \lim_{x \to c} f(u) = \lim_{u \to k} f(u)$$

Informal: If f(u) is close to L when u is close to k, and if u is close to k when x is close to c, then f(u) = f(g(x)) is close to L when x is close to c;
or, you can replace u -> k with x -> c in a limit provided x -> c implies u -> k.

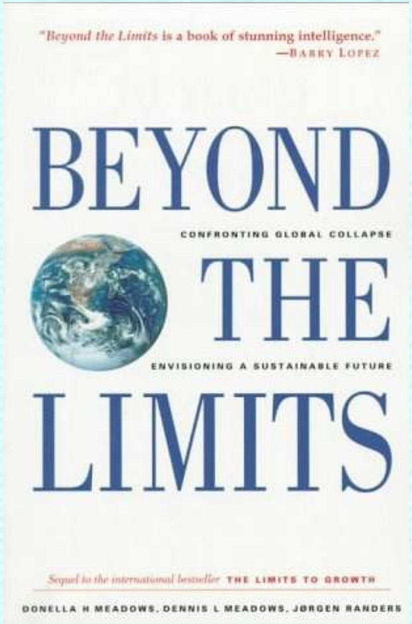
Various limits

Various limits



Various limits

Various limits

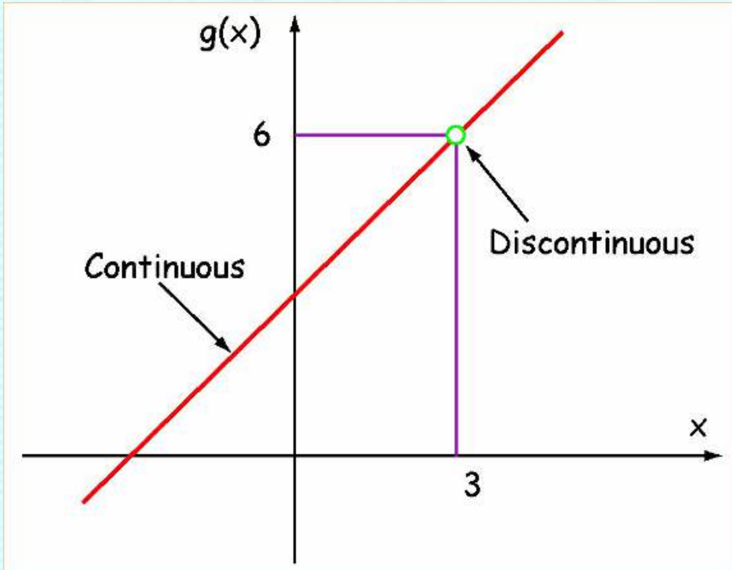


Various limits

Continuity

- A function such as $g(x) = (x^2 + 9) / (x-3)$ has a discontinuity at $x = 3$ because the denominator is zero there.

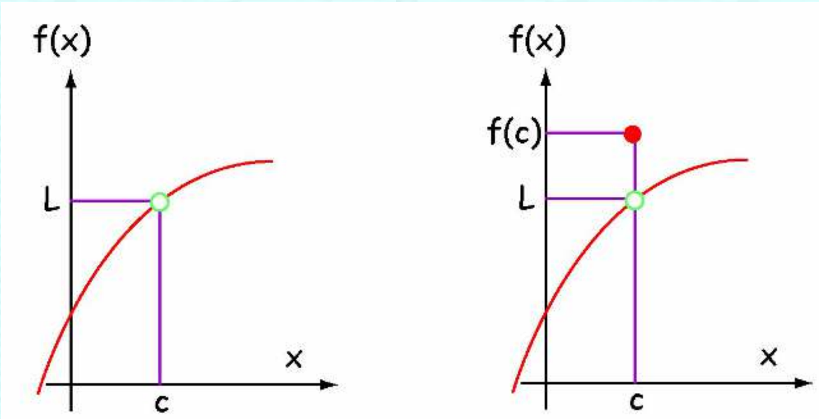
- It seems reasonable to say that the function is "continuous" everywhere else because the plot seems to have no other "gaps" or "jumps".



- Next we'll define continuity and use it in several ways.

Continuity

- These two functions have a limit as x approaches c.

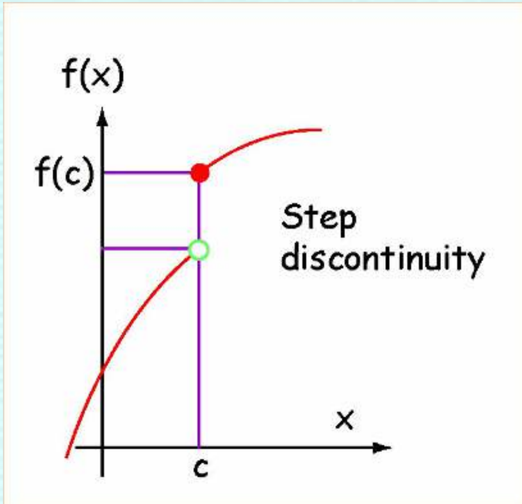


- But, the first is discontinuous at c because there is no value of f(c), while the second is discontinuous at c because f(c) doesn't equal L.

- The value of f(c) can be defined or redefined to make f continuous there.

Continuity

- This function has a step discontinuity at x = c.

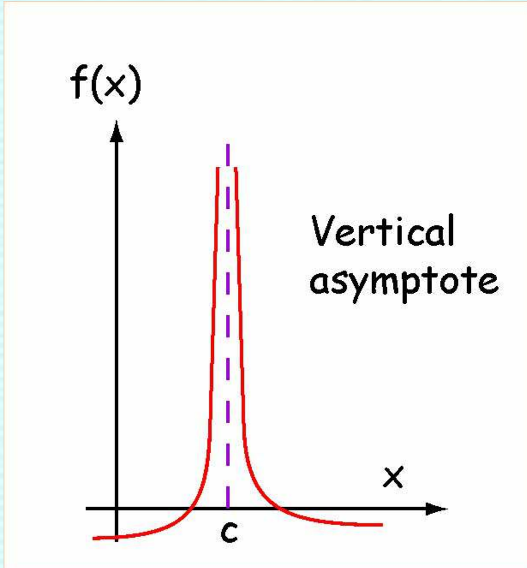


- Although there is a value f(c), f(x) approaches different values from the left of c and the right of c. Thus, there is no limit as x approaches c.

- A step discontinuity cannot be removed simply by redefining f(c).

Continuity

- This function is discontinuous at x = c because it has a vertical asymptote at c.

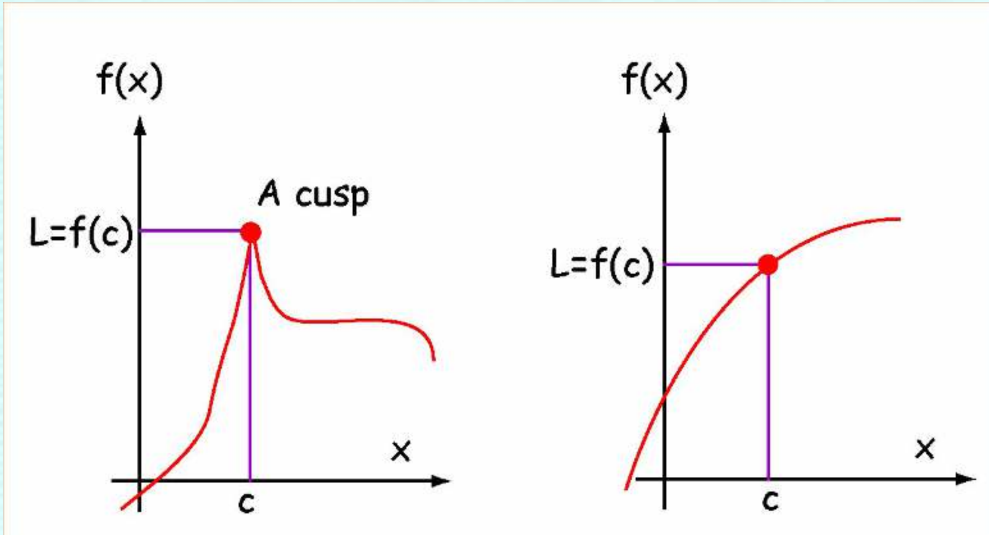


- There is neither a value of f(c) nor a finite limit of f(x) as x approaches c.

- Again, the discontinuity cannot be removed simply by redefining f(c).

Continuity

- These two functions are continuous at x = c.



- The value of f(c) equals the limit of f(x) as x approaches c.
  The branches of the plots are "connected" by f(c).

Definition

- A function f is continuous at x = c if and only if:

$$1)\ \ f(c) \text{ exists}$$

$$2)\ \lim_{x \to c} f(x) \text{ exists}$$

$$3)\ \lim_{x \to c} f(x) = f(c)$$

- If you can draw a function without lifting your pencil off the paper, then the function is continuous.

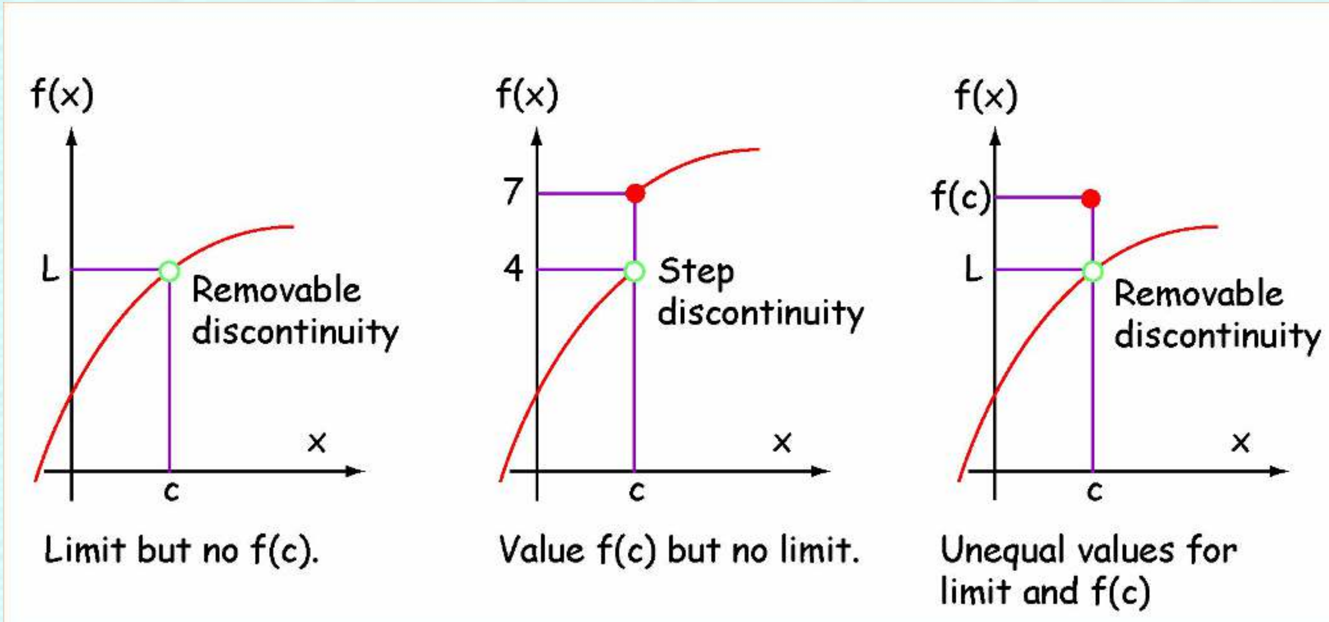
Cusps

- Note that a function can have a cusp (an abrupt change in direction) at x = c and still be continuous.

- A cusp is a point on a graph where the function is continuous but the derivative is discontinuous (a sharp point, like your bicuspids).

Continuity

- Why a function must satisfy all three parts of the continuity definition:



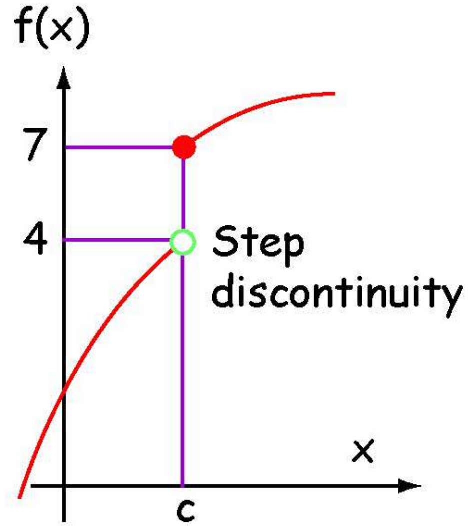
Continuity

- A step function illustrates the concept of a one-sided limit. If x is close to c on the left side, f(x) is close to 4. If x is close to c on the right, f(x) is close to 7.

- Symbols used for left and right limits:

$$\lim_{x \to c^-} \qquad \lim_{x \to c^+}$$

- Which means x -> c through the values of x on the negative and positive sides of c respectively.

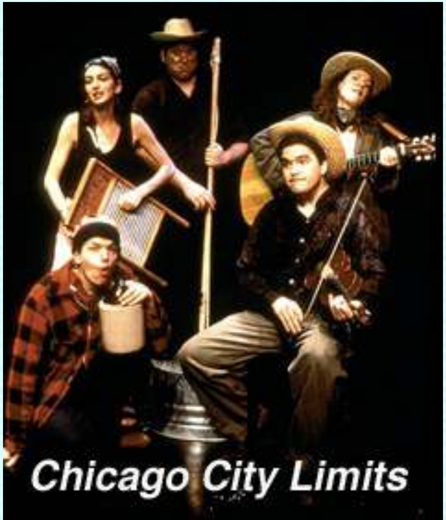


Value f(c) but no limit.

Continuity

- A function has a two-sided limit at x = c if both one-sided limits are equal.

$$L = \lim_{x \to c} \text{ if and only if } L = \lim_{x \to c^-} \text{ and } L = \lim_{x \to c^+}$$



Example

- Let the function be

$$f(x) = \begin{cases} 0.5x + 3 & \text{if } x < 2 \\ -x^2 + 6x - 2 & \text{if } x \geq 2 \end{cases}$$

Draw the graph,
find the left and right limits as x approaches 2,
and tell whether or not f(x) is continuous at x = 2.

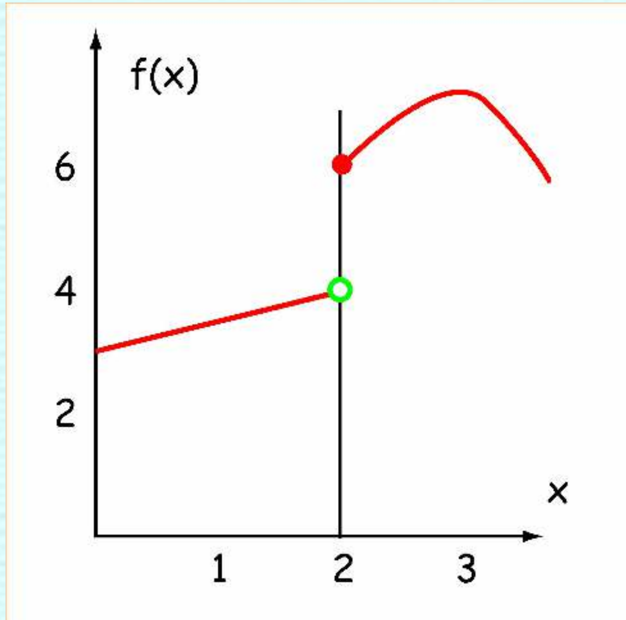
Continuity

- There is a step discontinuity at x = 2 because the two branches do not connect.

- The limit as x approaches 2 from the left is 4 because f(x) is close to 4 when x is close to, but less than 2.

- The limit as x approaches 2 from the right is 6 because f(x) is close to 6 when x is close to, but more than 2.

- The function is not continuous at x = 2 because f(x) has no limit as x approaches 2.



Playtime

- During your in-class problem solving session today you'll investigate the limits and continuity of some functions.