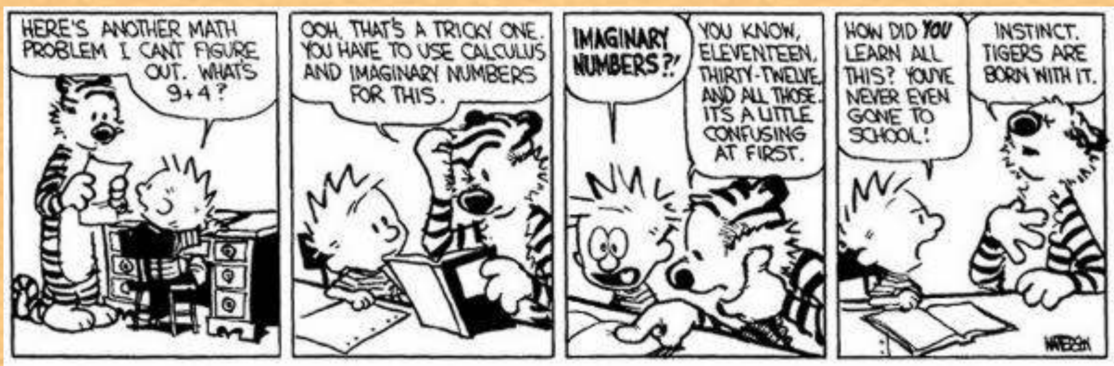

HERE'S ANOTHER MATH PROBLEM I CAN'T FIGURE OUT. WHAT'S 9+4?
OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE, AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.
HOW DID YOU LEARN ALL THIS? YOU'VE NEVER EVEN GONE TO SCHOOL!
INSTINCT. TIGERS ARE BORN WITH IT.
WATERSON

School of the Art Institute of Chicago

Calculus

Frank Timmes

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flash.uchicago.edu/~fxt/class_pages/class_calc.shtml

Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- www.friesian.com/calculus.htm
- archives.math.utk.edu/visual.calculus/1/continuous.7/
- www.infinite-image.net/

Class #5

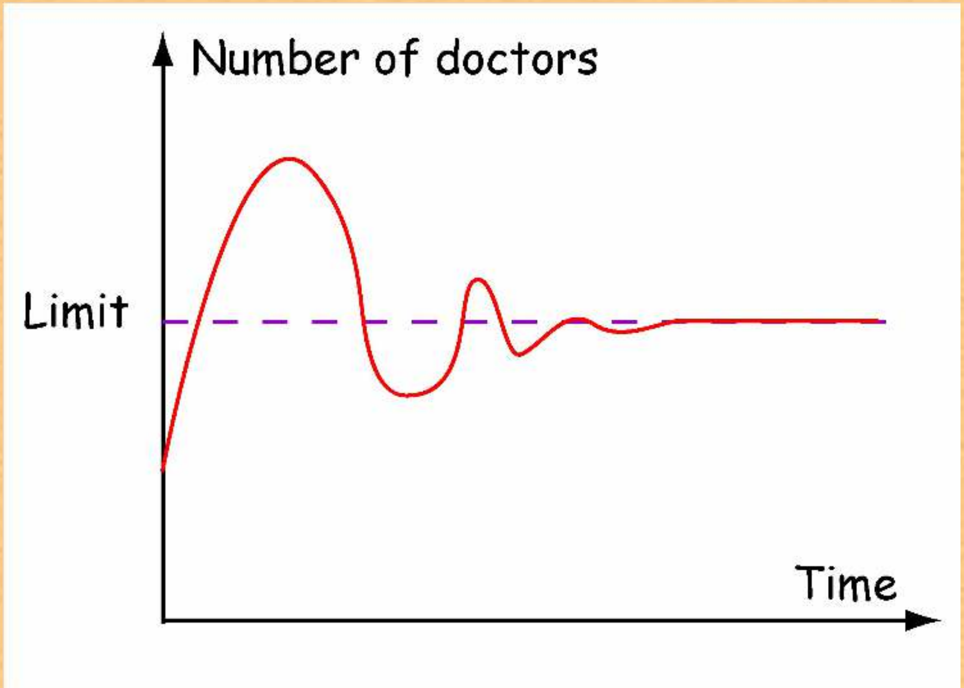
- Limits involving infinity
- Intermediate value theorem

Infinite limits

- Suppose there is an increase in the demand for doctors in a locale.
- The number of people who pursue that career will increase to meet the demand.
- After awhile, there may be too many doctors, causing the number of people who want to enter the medical profession to decrease.

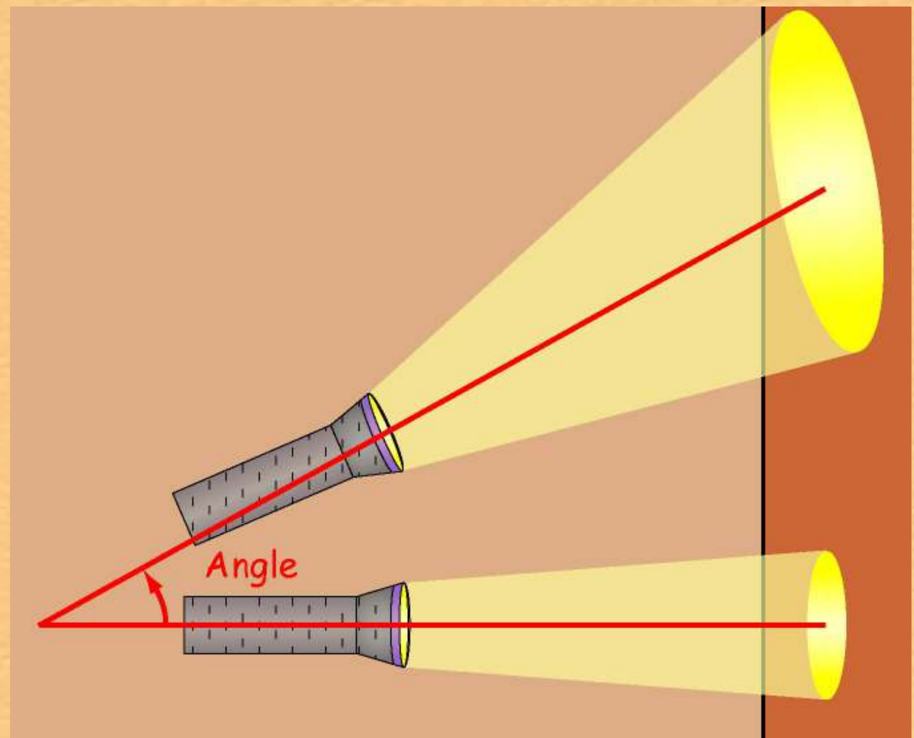
Infinite limits

- Eventually, the number of doctors stabilizes. This steady state value is called the limit of the number of doctors as time approaches infinity.



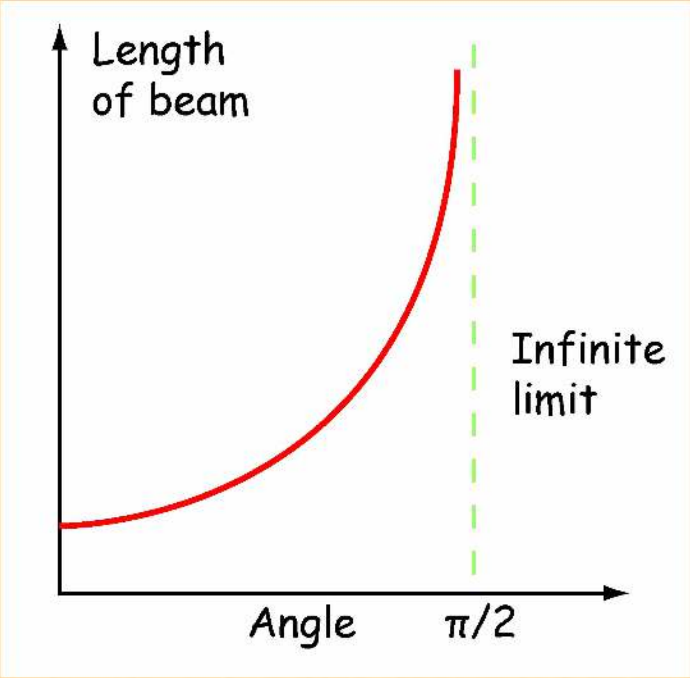
Infinite limits

- Another type of limit involving infinity can be visualized by pointing a flashlight straight at a wall.
- As we turn the flashlight upward, the length of the light beam increases.



Infinite limits

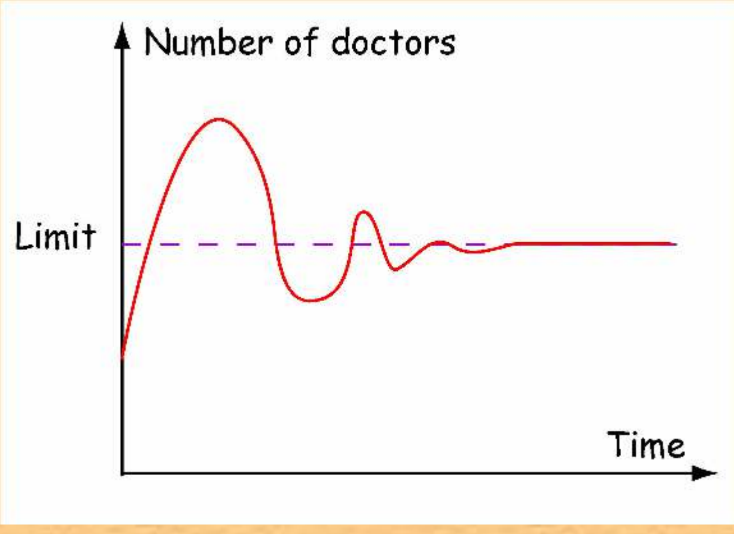
- When the angle is $\pi/2 = 90°$, the beam is parallel to the wall and its length becomes infinite.



Infinite limits

- The limit definition we've learned tells us what happens to the y-value of a function when x is kept close to a certain number c. But if x increases without bound, there isn't any number x can be kept close to.

- Suppose our doctor function can be made arbitrarily close to a given number, 100, just by making x large enough.



- Then 100 is the limit of the function as x approaches infinity.

Definition

$L = \lim_{x \to \infty} f(x)$ if and only if

for any number $\varepsilon > 0$,

there is a number $D > 0$ such that if $x > D$,

then $f(x)$ is within ε units of L.

- A similar definition holds for x approaching negative infinity.

Definition

- The same reasoning can be applied if the function becomes infinite as x approaches c, as in our flashlight example …

$\lim_{x \to c} f(x)$ is infinite if and only if

for any number $E > 0$,

there is a number $\delta > 0$ such that

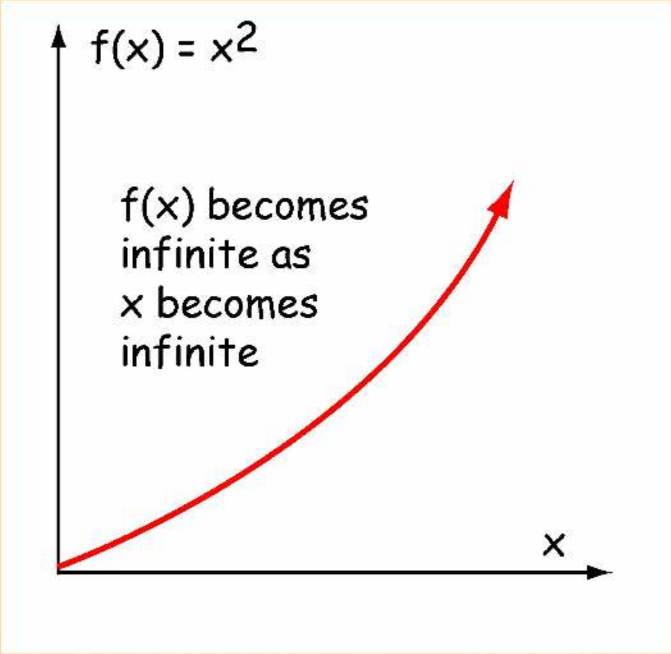
if x is within δ units of, but not equal to c,

then $|f(x)|$ is greater than E.

Infinite limits

- The graph of $f(x) = x^2$ displays this property of increasing without bound as x increases without bound.

- In plain English, "If you can make y as big as you want by making x big enough, then the limit of f(x) is infinite as x approaches infinity".



Definition

- If f(x) becomes infinite as x becomes infinite, both modifications to the usual definition of limit are used.

$\lim_{x \to c} f(x)$ is infinite if and only if

for any number $E > 0$,

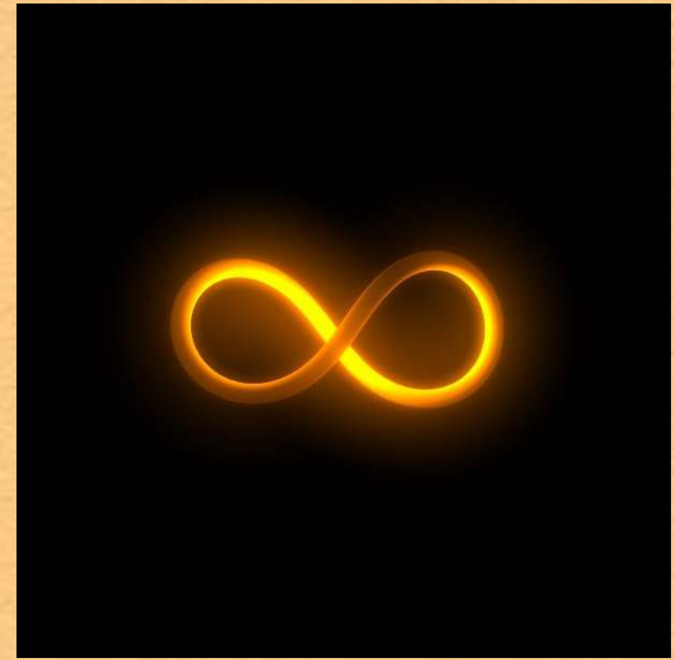
there is a number $D > 0$ such that if $x > D$,

then $|f(x)| > E$.

- Similar definitions holds for x approaching negative infinity.

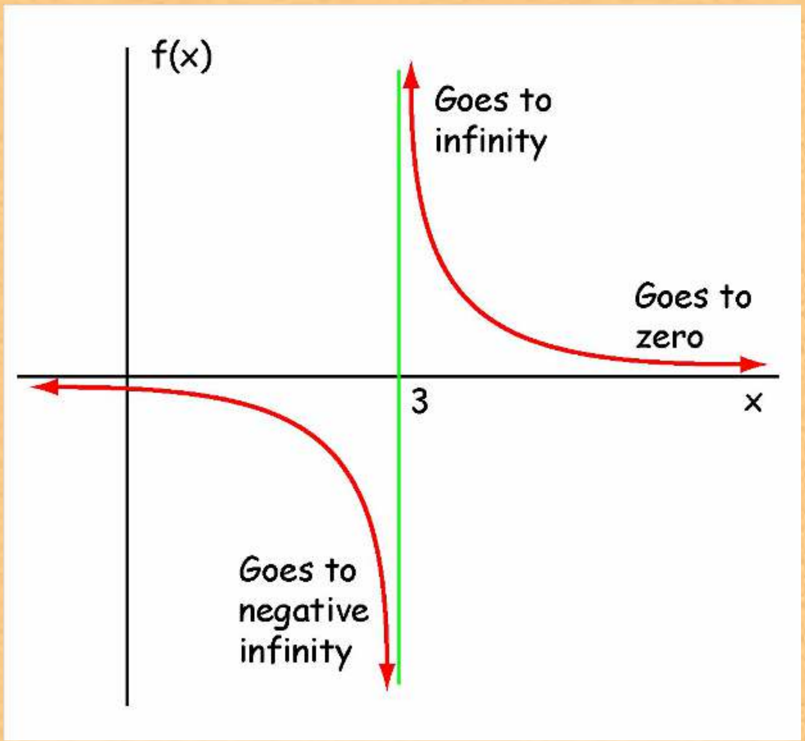
Zero and Infinity

- Two properties come directly from the previous definitions.
  They concern what happens to the reciprocal of a function if the function approaches either zero or infinity.



Zero and Infinity

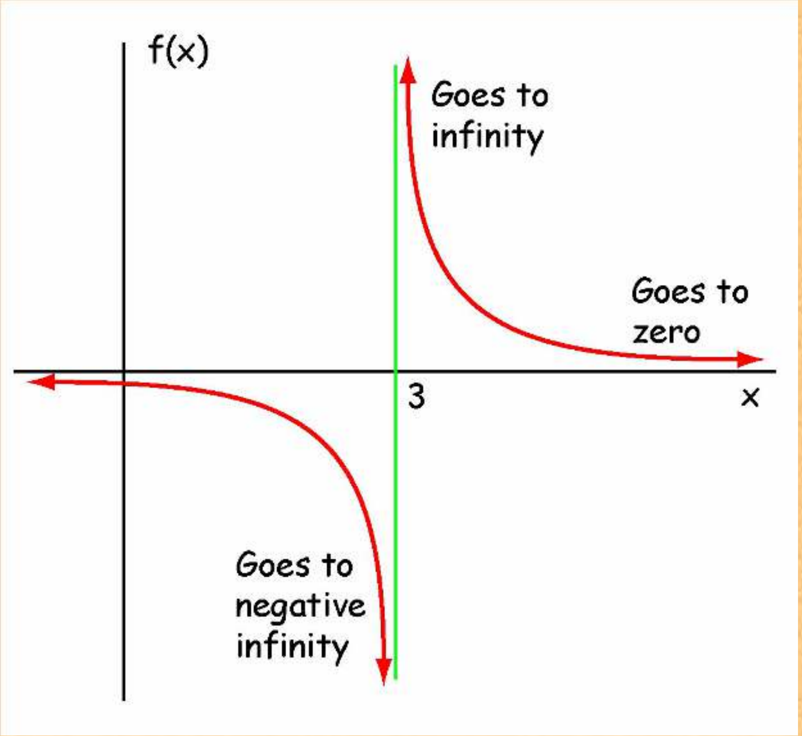
- Suppose $f(x) = 1/(x - 3)$.
- As x approaches 3, the denominator goes to zero.



Zero and Infinity

- The function becomes infinitely large if x is to the right of 3, and negative-infinitely large to the left of 3.

- If x approaches infinity or negative infinity, then the denominator becomes infinite and the function approaches zero.



Definition

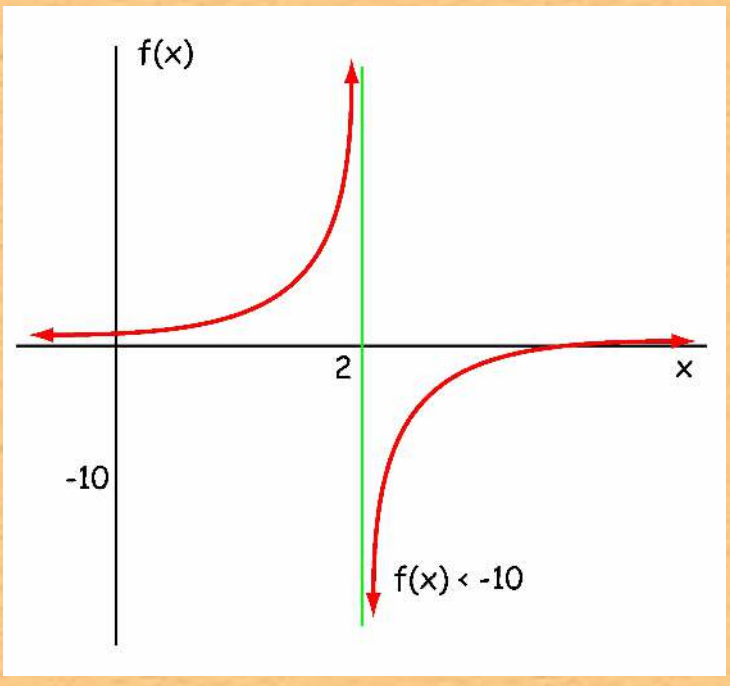
$$\text{if } f(x) = \frac{1}{g(x)} \text{ and } \lim_{x \to c} g(x) = 0, \text{ then } \lim_{x \to c} f(x) = \infty$$

$$\text{if } f(x) = \frac{1}{g(x)} \text{ and } \lim_{x \to c} g(x) = \infty, \text{ then } \lim_{x \to c} f(x) = 0$$

- The same properties apply if x approaches infinity.
- Informally, this says 1/zero is infinite and 1/infinity is zero.

Example

- For f(x) = (x - 4)/(x - 2) show that the limit of f(x) as x approaches 2 from the positive side is negative infinity.
- The graph shows there is a vertical asymptote at x = 2 because of division by zero, so the limit is infinite.
- Since f(x) has the form negative/ positive when x is close to 2 on the positive side, the limit is negative infinity.



Example

- Find a value of δ such that $|f(x)| > 10$ whenever x is within δ units of 2 on the positive side.

$$-10 = \frac{x-4}{x-2}$$

$$-10x + 20 = x - 4$$

$$11x = 24$$

$$x = 24/11$$

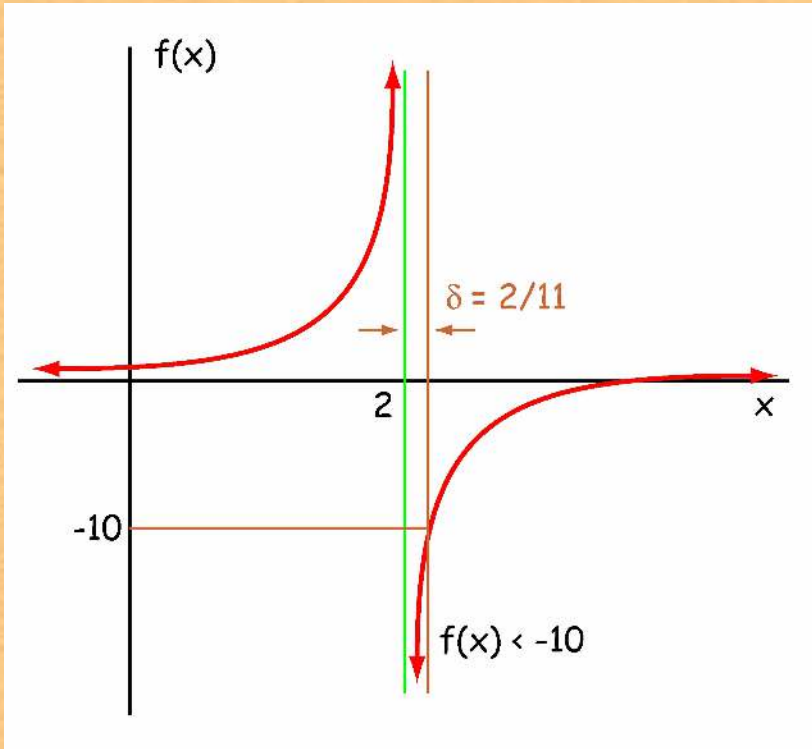
$$\delta = x - 2$$

$$= 24/11 \; - \; 22/11$$

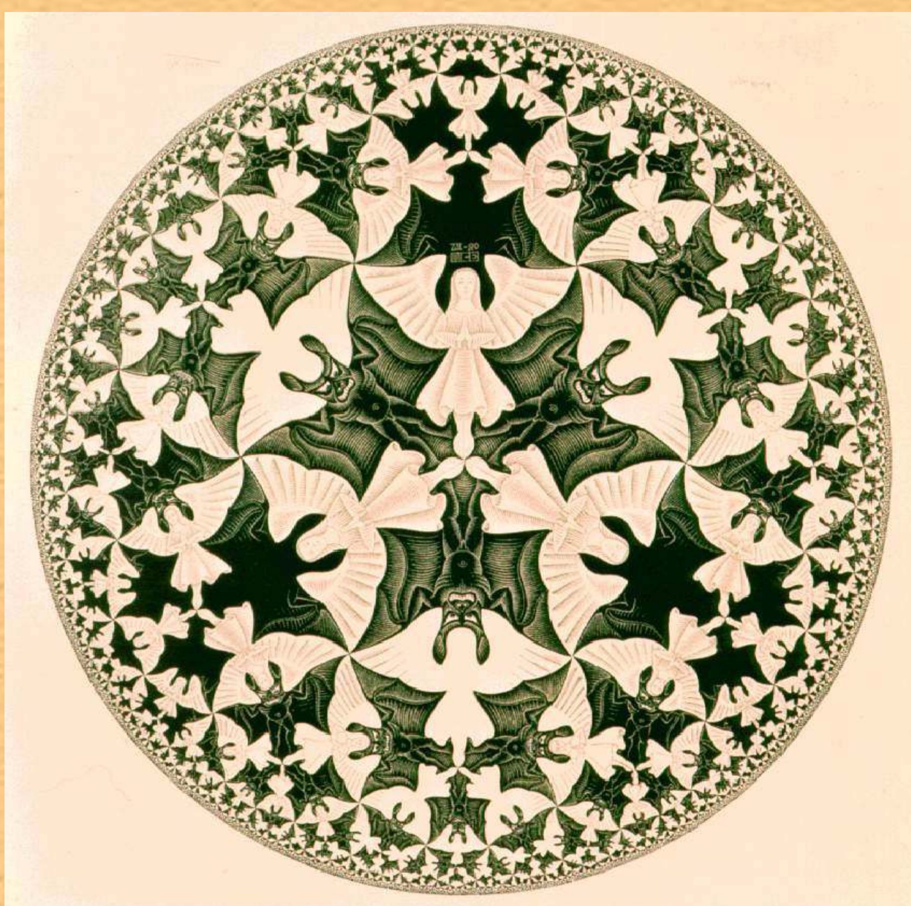
$$= 2/11$$

Example

- Thus, $|f(x)| > 10$ when x is within 2/11 units to the right of 2.

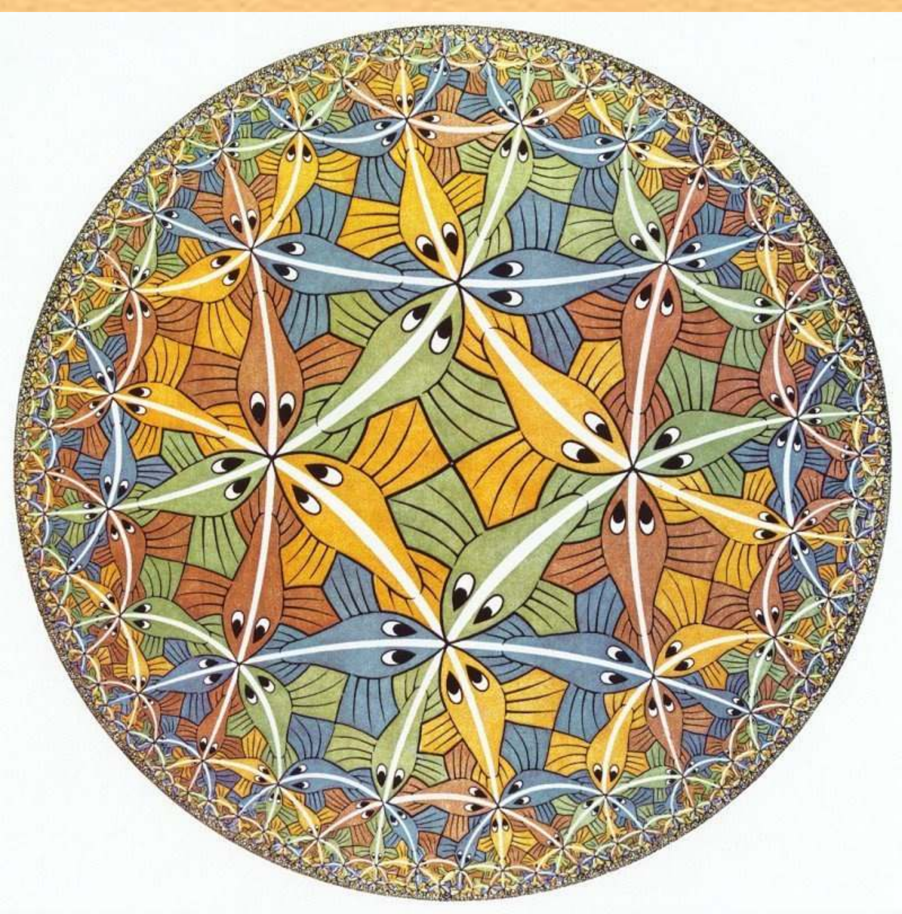


Infinity



Circle limit III,
1959, M.C. Escher

Infinity



Circle limit IV,
1959, M.C. Escher

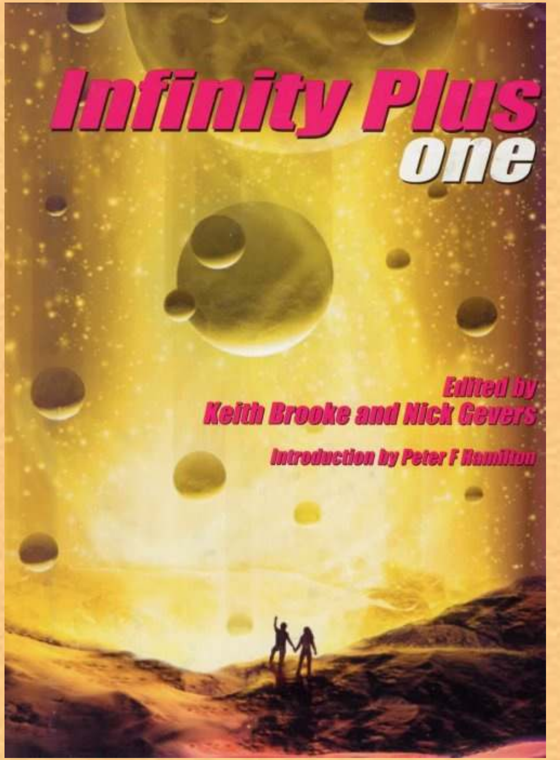
Infinity

Infinity cow,
2001, Chicago Cows

Infinity

Infinity Artworks

Infinity


Infinity Plus
one
Edited by
Keith Brooke and Nick Gevers
Introduction by Peter F Hamilton

In between

- Suppose you try to and find a solution of $x^3 = 6$ by making a plot of $y = x^3$. A cursor never quite hits a value of x that makes y equal to exactly six.

- That's because any grapher plots discrete points that represent only approximately the continuous graph.

- However, because $y = x^3$ is continuous, there really is a value of x (an irrational number) which, when cubed, gives exactly six.

In between

- The property of continuous functions that guarantees there is an exact value is called the intermediate value theorem.

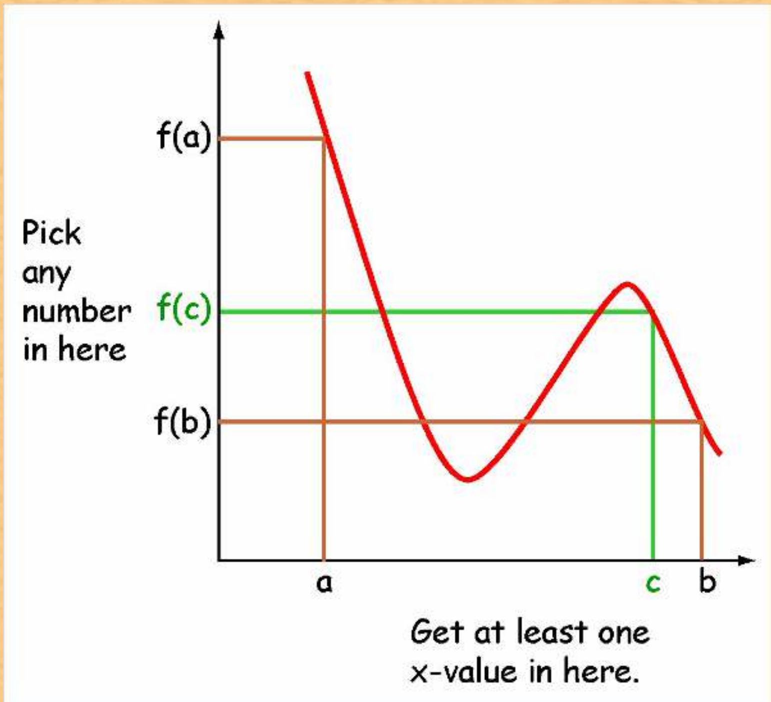
Intermediate value theorem:

If a function is continuous for all x in the closed interval $[a,b]$
and y is a number between $f(a)$ and $f(b)$,
then there is a number $x = c$ in (a,b) for which $f(c) = y$.

- Informally it says that if you pick a value of y between any two values of $f(x)$, there is an x-value in the domain that gives exactly that y value for $f(x)$.

In between

- Pick any y between f(a) and f(b).



- If f(x) is continuous, you can go from the y value over to the curve, then go down to the x-axis to find the corresponding value of x.

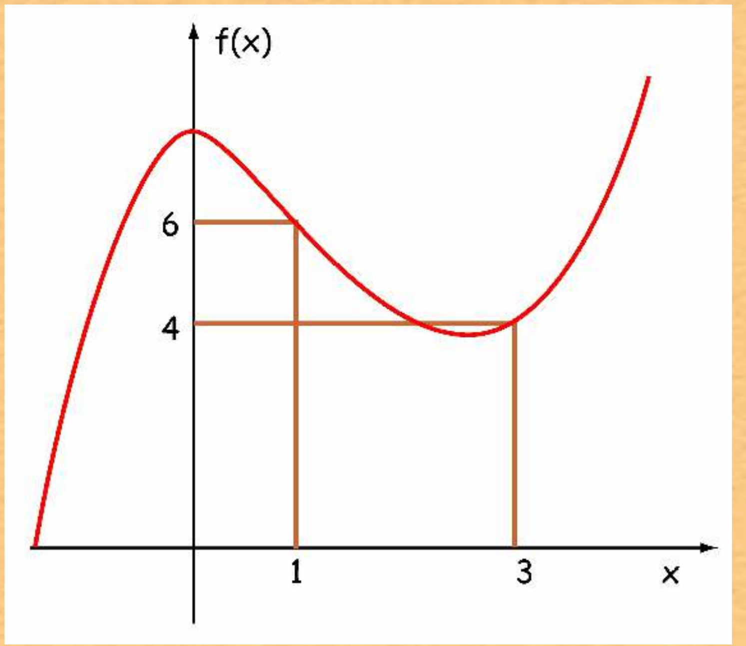
In between

- Proof of the intermediate value theorem relies on the completeness axiom.

- This axiom says that there is a real number corresponding to every point on the number line. Thus, the set of real numbers is “complete”. It doesn’t have any “holes” as does, for example, the set of rational numbers.

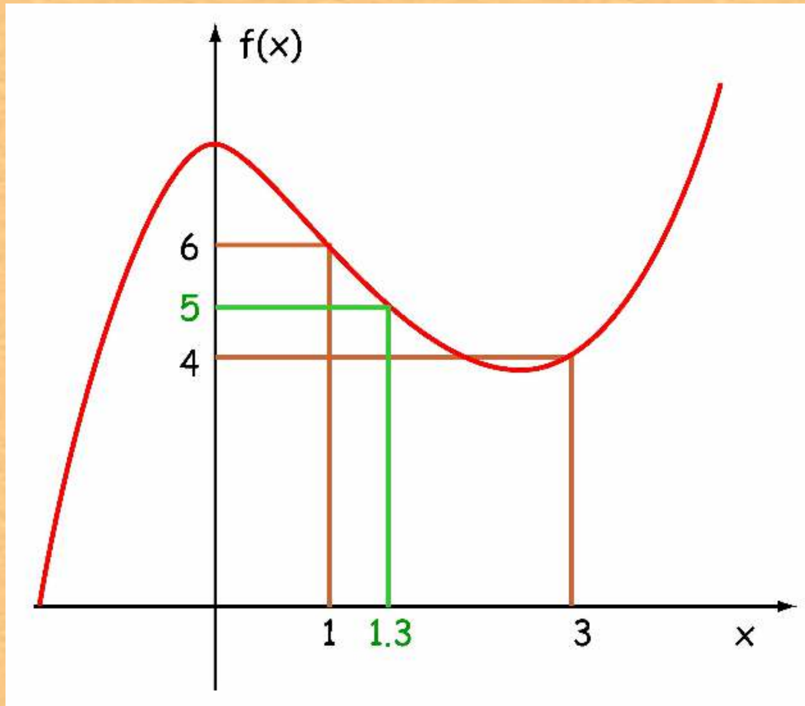
Example

- If $f(x) = x^3 - 4x^2 + 2x + 7$,
  find, approximately, a value of c between 1 and 3 for which $f(c) = 5$.
  Prove that there is a value of c for which $f(c)$ is exactly 5.



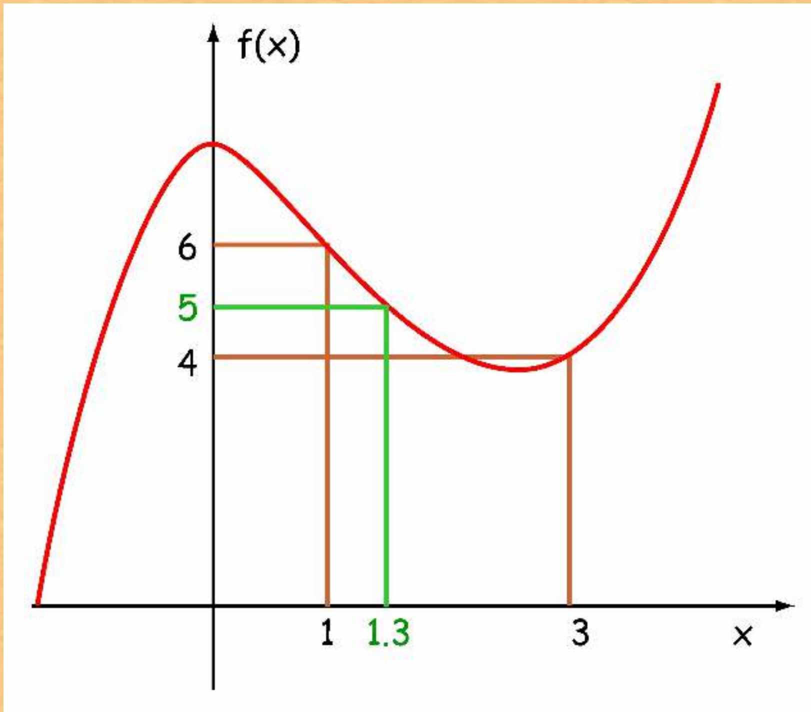
Example

- A line at y = 5 intersects the graph at three points. The one between 1 and 3 is about 1.3.



Example

- $f(x)$ is a polynomial, and thus continuous. Since 5 is between $f(1) = 6$ and $f(3) = 4$, the intermediate value theorem proves there is an exact value of c such that $f(c) = 5$.



Playtime

- During your in-class problem solving session today you'll look at limits involving zero and infinity and use the intermediate value theorem.