Students attend our lectures, not because the mathematics we teach “makes lots of fun” for us, but because they believe they can learn some essential knowledge from us. And each of our students has only one life to live. We should therefore be able to justify ourselves to our listeners with respect to what we teach them.

H. Behnke

School of the Art Institute of Chicago

Calculus

Frank Timmes

ftimmes@artic.edu

flash.uchicago.edu/~fxt/class_pages/class_calc.shtml

Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of

Sites of the Week

- hyperphysics.phy-astr.gsu.edu/hbase/deriv.html
- www2.ncsu.edu/unity/lockers/users/f/felder/public/kenny/papers/dx.html

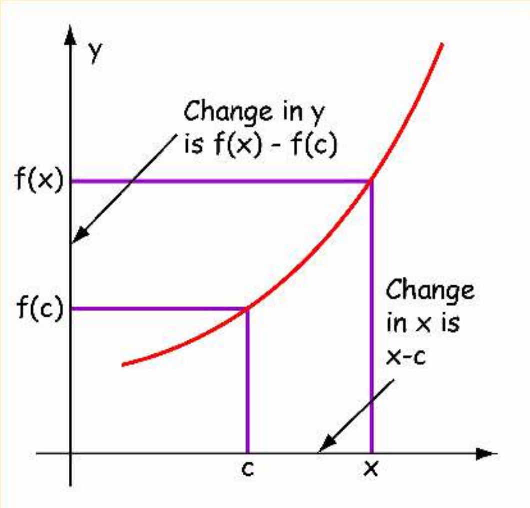
Class #6

- Definition of a derivative
- Numerical and graphical derivatives
- Exact derivatives of polynomials

Differences

- We have been finding average rates of change by taking a change in x and dividing it into the corresponding change in y.

- The derivative is the function
  we get by taking the limit as the change in x approaches zero.



Differences

- One way to write the definition of a derivative at point c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

- If the function’s name is f, then the symbol f’ (pronounced “f prime”) is often used for the derivative function.

- The symbol shows there is a relationship between the original function and the function “derived” from it (hence the name derivative), indicating that its an instantaneous rate of change.

Example

- If $f(x) = x^2 - 3x - 4$,
  find the derivative at $x = 5$,
  $f'(5)$.

$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5}$$

$$f'(5) = \lim_{x \to 5} \frac{(x^2 - 3x - 4) - f(5)}{x - 5}$$

$$f'(5) = \lim_{x \to 5} \frac{x^2 - 3x - 4 - 6}{x - 5}$$

$$f'(5) = \lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5}$$

$$f'(5) = \lim_{x \to 5} \frac{(x - 5)(x + 2)}{x - 5}$$

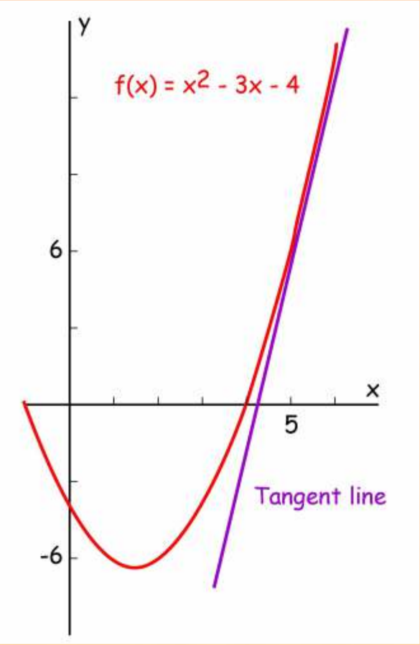
Whew! Hard work.

$$f'(5) = \lim_{x \to 5} (x + 2) = 5 + 2 = 7$$

Tangent line

- The derivative has a geometrical meaning: its the slope of the tangent line, the slope of the line that just touches the curve.

- Both derivative and the slope of the tangent line equal the instantaneous rate of change.



Tangent line

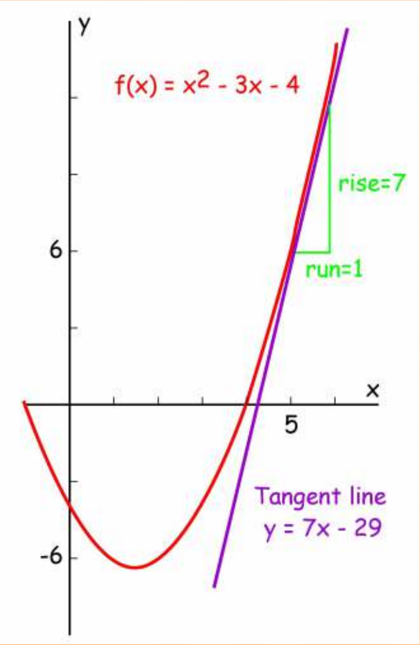
- We know that $f(5) = 6$ and $f'(5) = 7$. The equation of the line running through this point with this slope is:

$$y = m \cdot x + b$$

$$6 = 7 \cdot 5 + b$$

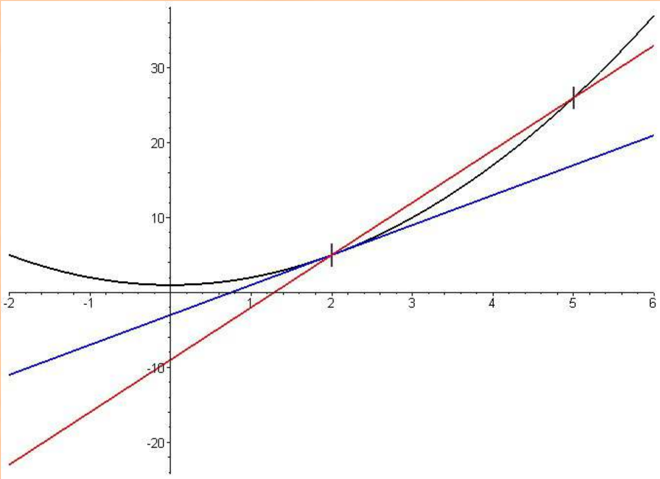
$$b = -29$$

$$y = 7x - 29$$



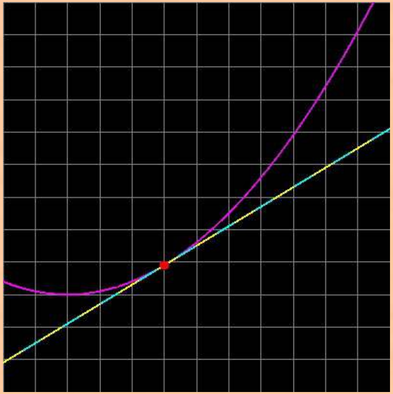
Tangent line

- The tangent line is the limit of secant lines.
  The instantaneous rate is the limit of the average rates.



Tangent line

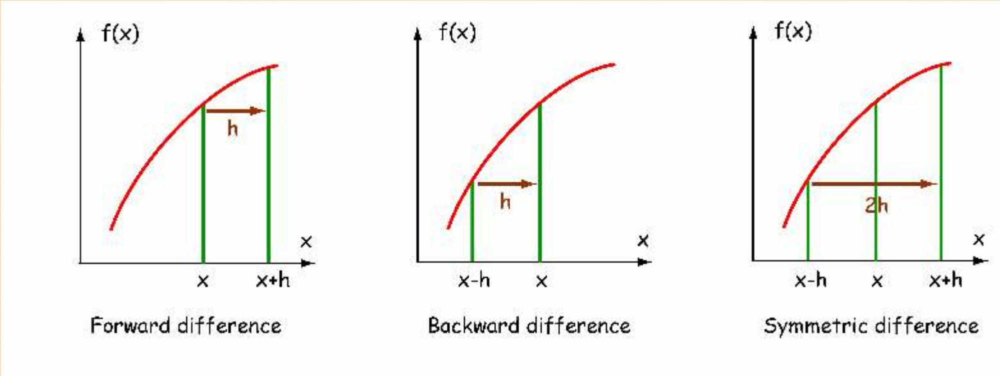
- At a small enough scale, all curves are straight lines.



Numerical derivatives

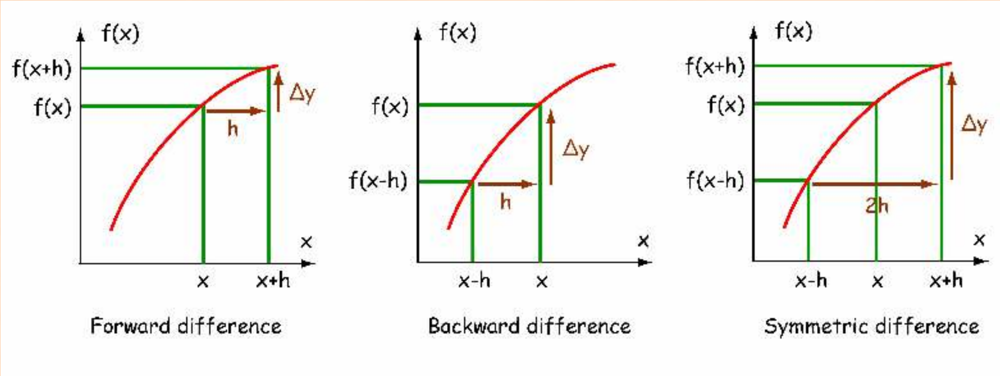
- Some graphers calculate numerical derivatives by using difference quotients, just as we have been doing.

- First, points on the x-axis a distance h (“horizontal”) from x are chosen.



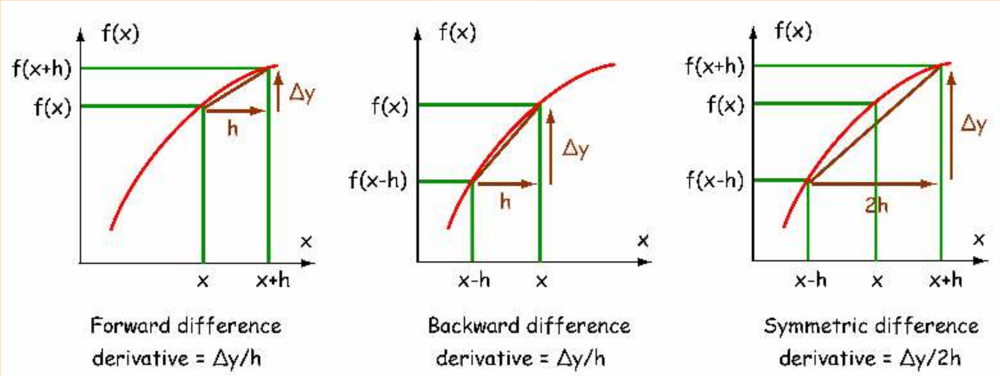
Numerical derivatives

- The corresponding difference in the y-values, Δy ("delta y") are found …



Numerical derivatives

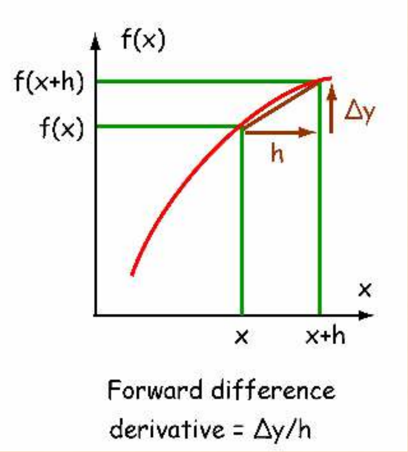
- … and divided to find an approximation to the instantaneous rate of change.



Numerical derivatives

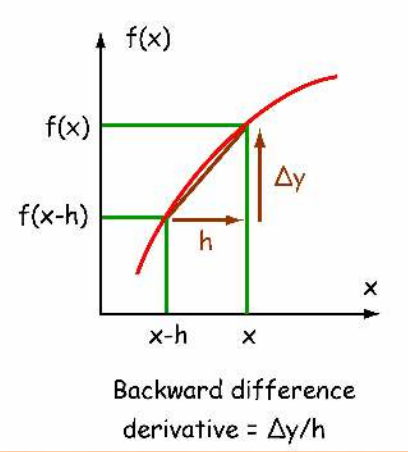
- Estimate the derivative of x^3 at $x = 2$ with $h = 0.01$

Forward difference: $[f(2.01) - f(2)] / 0.01 = (2.01^3 - 2^3)/0.01 =$
12.0601



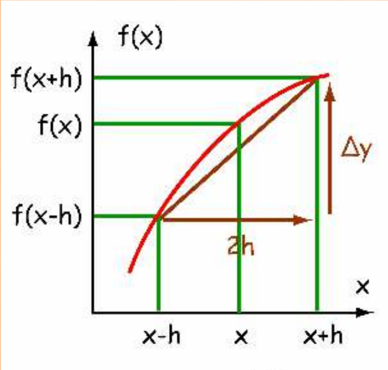
Numerical derivatives

Backward difference: [f(2) - f(1.99)] /0.01 = $(1.99^3 - 2^3)/0.01$
= 11.9401



Numerical derivatives

Symmetric difference: [f(2.01] - f(1.99)] /0.02 = 12.0001

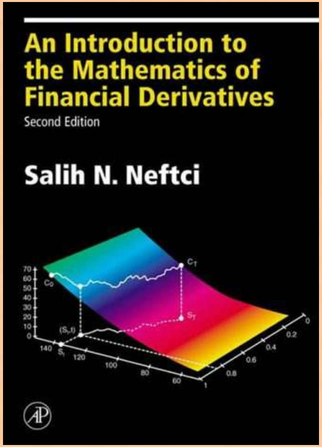


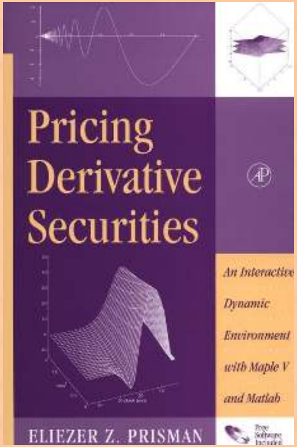
Symmetric difference derivative = Δy/2h

General derivatives

- We've been calculating the derivative of a *given* function at one point, x = c. Now its time to turn our attention to finding the derivative for all values of x.

- We seek a new function whose values are the derivative of the given function.

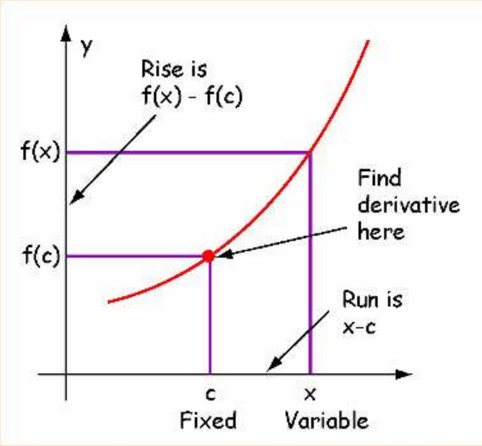




Derivative definitions

- Let's re-cast our definition for the derivative of a function at a point x = c
into another form that is more suitable for a general derivative.

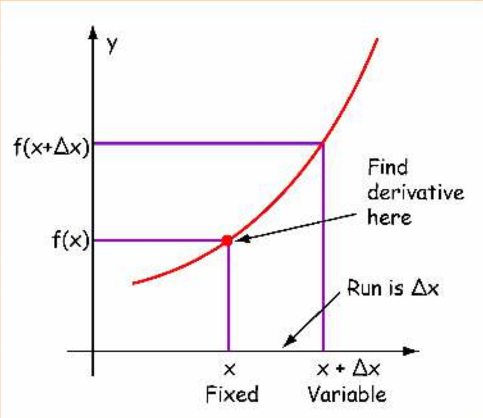
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$



Derivative definitions

- Replace c with “x” and x with “x + Δx”.
  The change in x-values thus takes the simpler form Δx ,“delta x.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

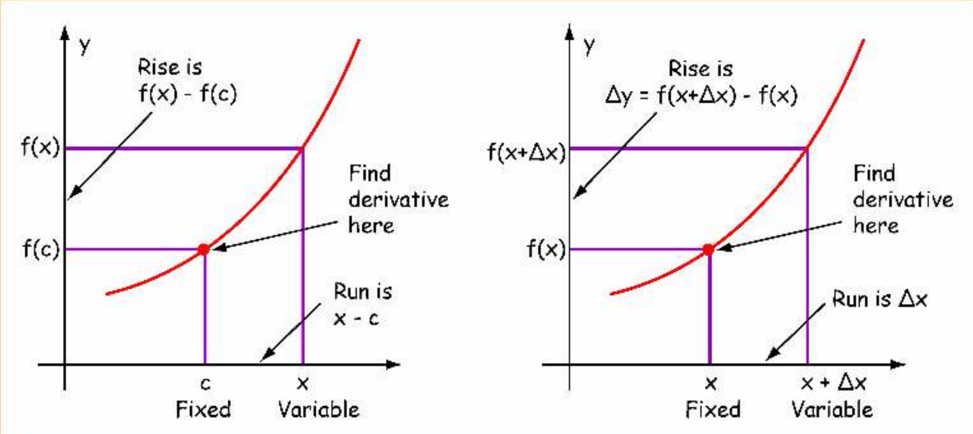


$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Derivative definitions

- The rise, Δy, is $f(x + \Delta x) - f(x)$

- As x approaches c in the figure on the left, Δx approaches zero in the figure on the right.



Derivative definitions

- So, we have the following equivalent definition of a derivative:

$$f' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

- Sometimes the single letter "h" (horizontal) is used in place of Δx to make the algebra easier to write, so we've included that form as well.

Example

- For $f(x) = x^3$, use the definition of a derivative to find an equation for $f'(x)$

$$f' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2 + 0 + 0$$

$$= 3x^2$$

Polynomial derivatives

- Note the relation between the function x^3 and its derivative $3x^2$. The coefficient, 3, is the original exponent, and the power, 2, is one less than the original exponent.

The general derivative of a power function:

If $f(x) = x^n$, then $f'(x) = n\,x^{n-1}$, where n is a constant.

- To differentiate the power function, $f(x) = x^n$, multiply by the old exponent, n, then reduce the exponent by 1 to get the new exponent.

Derivative properties

- To find a formula for differentiating a linear combination of power functions, we'll need a few properties of differentiation.

If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$

The derivative of a sum equals the sum of the derivatives.
Differentiation distributes over addition.

If $f(x) = k \cdot g(x)$, then $f'(x) = k \cdot g'(x)$, where k is a constant.
The derivative of a constant times a function equals the constant times the derivative of the function.

Derivative properties

If $f(x) = C$, where C is a constant, then $f'(x) = 0$

Constants don't change, so their derivative is zero.

Examples

- If $f(x) = 2x^5 + 9x^2$, then $f'(x) = 2 \cdot 5x^4 + 9 \cdot 2x =$
  $10x^4 + 18x$

- If $f(x) = ax^2 + bx + c$, then $f'(x) =$
  $2ax + b$

- If $f(x) = mx + b$, then $f'(x) =$
  m

- If $f(x) =$ constant, then $f'(x) = 0$

- A whole lot easier than differentiating from the definition!

Terminology

- If $y = f(x)$, then instead of writing $f'(x)$, you can write any of the following:

  y', pronounced “y prime” (a short form of $f'(x)$)

  dy/dx, pronounced “dee y, dee x” (a single symbol, not a fraction)

  $d/dx\,(y)$, pronounced “dee, dee x, of y (an operation done on y)

Terminology

- The symbol dy/dx comes from the difference quotient $\Delta y/\Delta x$. It means that the limit is to be taken as both Δy and Δx go to zero.

- For the time being, regard dy/dx as a single symbol that cannot be taken apart - avoid saying “dy over dx”. Later on we’ll see that dy and dx are called differentials.

- The symbol d/dx is an operator than acts on an expression, similar to sin x or log x^2. It tells you to take the derivative of something with respect to x.

Examples

- If $y = 7x^{-4/5}$, find dy/dx.

  $y' = 7 \cdot (-4/5)\, x^{-4/5 - 1} = -28/5\, x^{-9/5}$

- If $y = 7^5$, find y'.

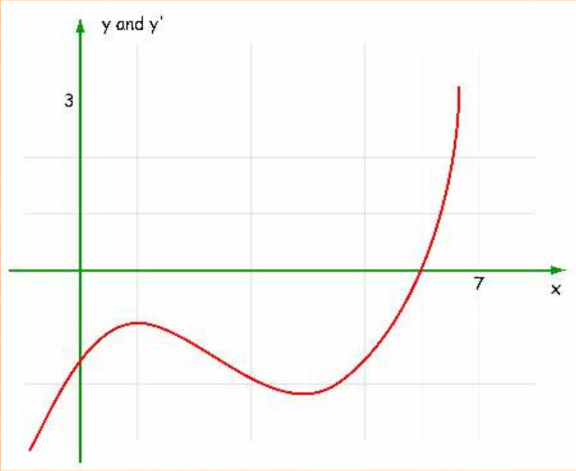
  $y' = 0$ because 7^5 is a constant.

Derivatives by eye

- One can graph the derivative of a function by looking at a graph of the function.

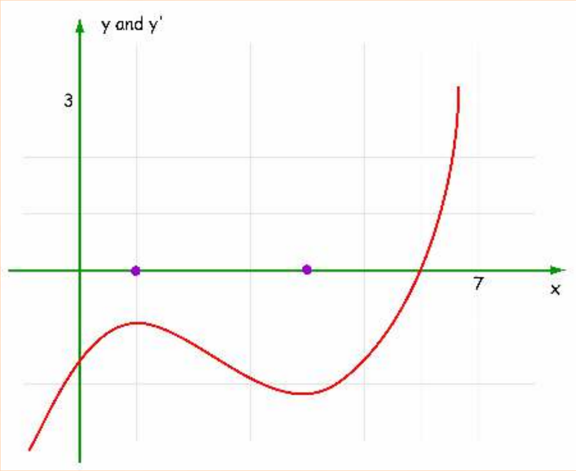
- At a high or low point in the function graph, the derivative will be zero.
  Why?

- If the graph is going up, the derivative will be positive and so on.



Derivatives by eye

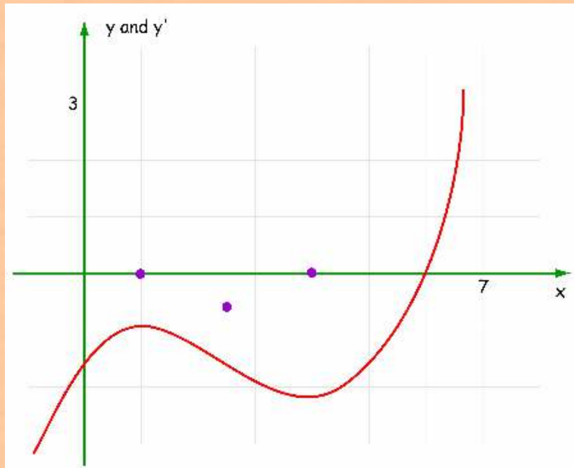
- At x = 1 and x = 4, the function has leveled off.
  Thus, it is not changing, its tangent line is horizontal, and its derivative is zero.
  Mark these two points.



Derivatives by eye

- Between x = 1 and x = 4, the function is decreasing (negative derivative).

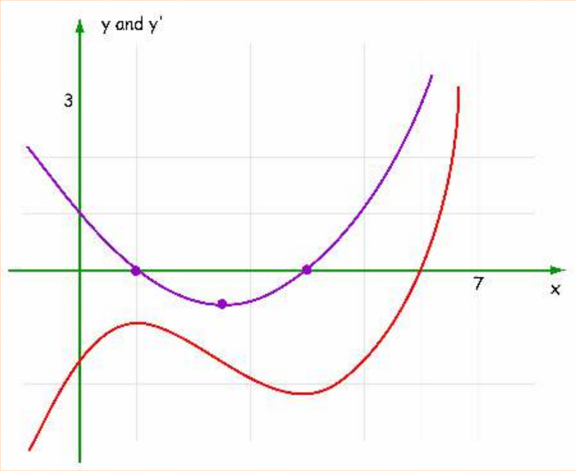
- The greatest downward slope occurs around 2.5. Mark a point below the x-axis around x = 2.5.



Derivatives by eye

- Above x = 4 the graph slopes up at an increasing rate.
  Draw the derivative curve as being positive and increasing.

- Below x = 1 the graph also slopes up, but at a smaller rate.
  Draw the derivative curve as being positive and decreasing.



Playtime

- During your in-class problem solving session today you'll find some derivatives from the definition, calculate derivatives of polynomials using the shortcuts, and sketch a few derivative functions.