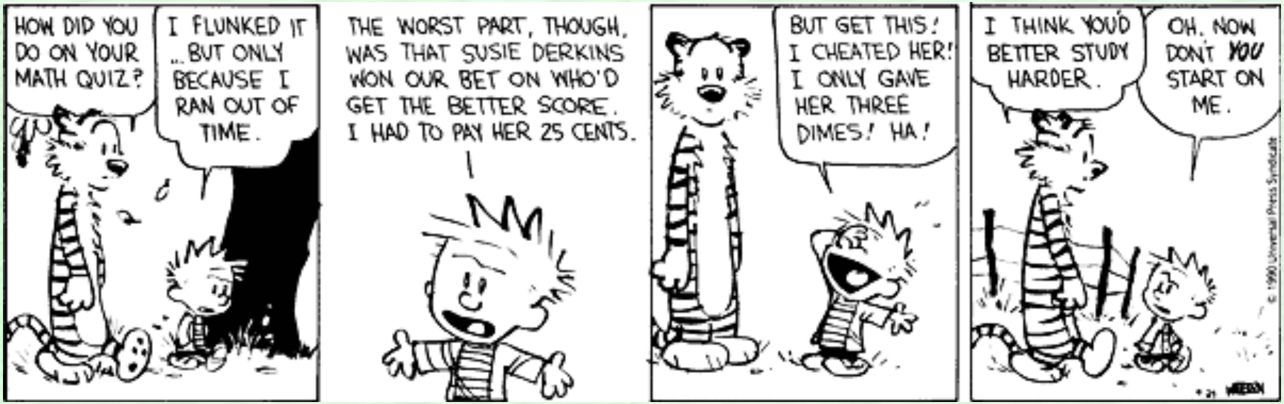
HOW DID YOU DO ON YOUR MATH QUIZ?
I FLUNKED IT ...BUT ONLY BECAUSE I RAN OUT OF TIME.
THE WORST PART, THOUGH, WAS THAT SUSIE DERKINS WON OUR BET ON WHO'D GET THE BETTER SCORE. I HAD TO PAY HER 25 CENTS.
BUT GET THIS! I CHEATED HER! I ONLY GAVE HER THREE DIMES! HA!
I THINK YOU'D BETTER STUDY HARDER.
OH, NOW DON'T YOU START ON ME.
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WATTERSON

School of the Art Institute of Chicago

Calculus

Frank Timmes

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flash.uchicago.edu/~fxt/class_pages/class_calc.shtml

Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- www.sosmath.com/calculus/diff/der04/der04.html
- archives.math.utk.edu/visual.calculus/2/chain_rule.4/
- library.thinkquest.org/10030/calcucon.htm

Class #7

- Distance, speed, acceleration
- Sine and cosine derivatives
- Chain rule

Need for speed

- We've found the speed or acceleration of a moving object at a given point several times in this course.

- Now that we know how to find the exact derivative of polynomials, $d(y^n)/dt = n \cdot y^{n-1}$, we can get an equation for the velocity or acceleration if we have an equation for the position of an object.

Need for speed

- Suppose a soccer ball is kicked into the air. As it rises and falls, its distance above the ground is a function of the number of seconds since it was booted.

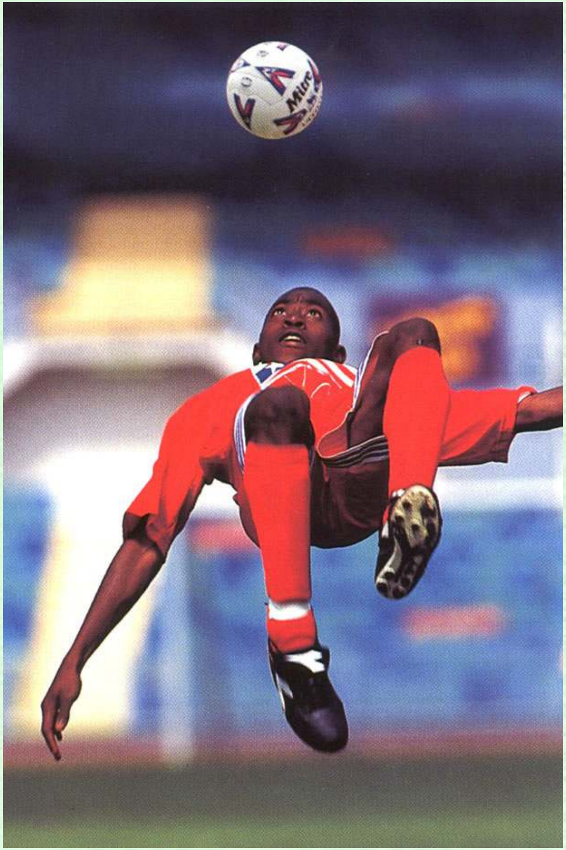
Need for speed

- Suppose experiment finds that $y = -16t^2 + 37t + 3$, where y is the height above the ground in feet and t is the number of seconds elapsed since the kick.

Need for speed

- Because speed is an instantaneous rate of change, it is a derivative:
  $v = dy/dt$
  $= d(-16t^2 + 37t + 3)/dt$
  $= -32t + 37$ ft/sec.

- The dy/dt symbol helps you remember the units of speed. y is in feet and t is in seconds, so dy/dt is in feet/sec.



Need for speed

- The instantaneous rate of change of speed is an acceleration.
  It is the derivative of the speed: $a = dv/dt = d(-32t + 37)/dt = -32$ feet/sec^2

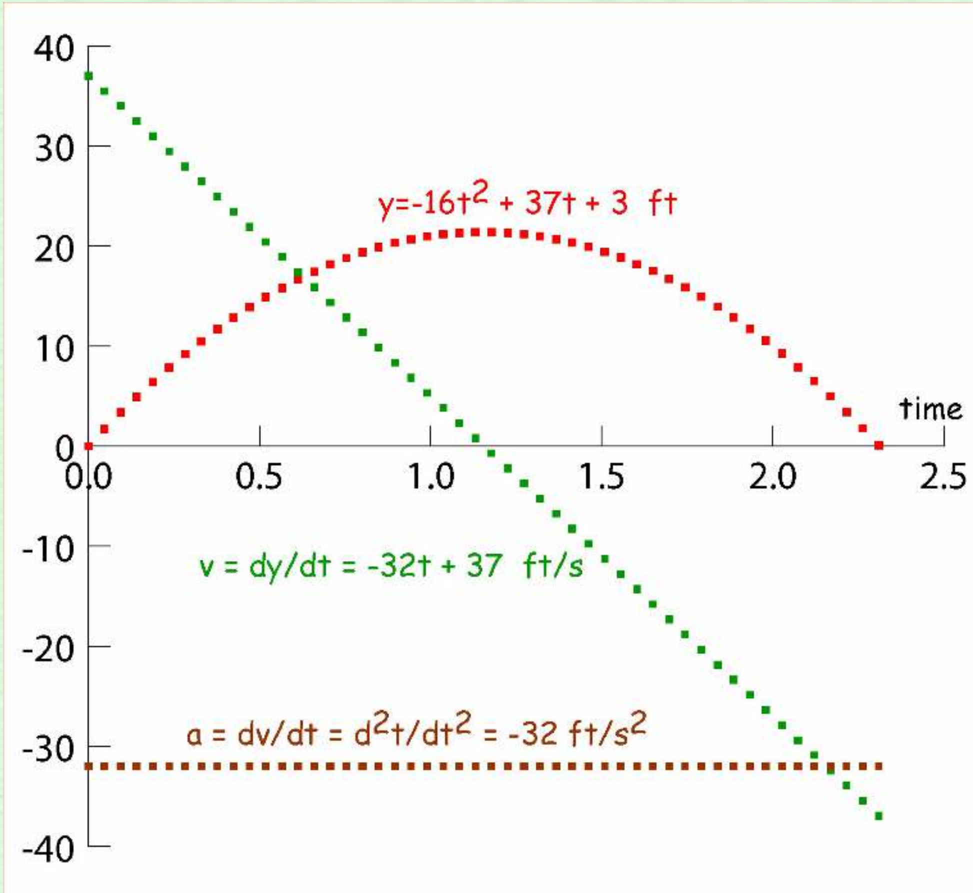
- The dv/dt symbol helps you remember the units of acceleration.
  v is in feet/sec and t is in seconds,
  so dv/dt is in feet/sec/sec = feet/sec^2.

Need for speed

- The negative acceleration means the speed is always decreasing;
  from 5 ft/sec at t = 1 sec
  to -27 ft/sec at t = 5 sec.

- Note that the acceleration is constant, -32 feet/sec^2, for any object acted on by gravity near the earth's surface.

Need for speed



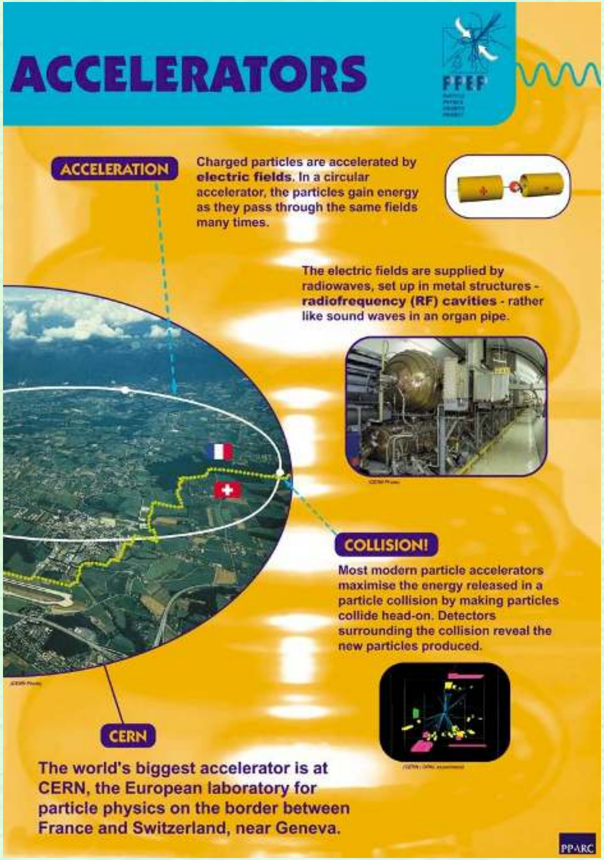
Second derivatives

- The acceleration of a moving object is the derivative of a derivative, and it's called a second derivative.

- The symbol for the second derivative of y with respect to t is d^2y/dt^2, pronounced "dee squared y, dee t squared".

Second derivatives

- The symbol d^2y/dt^2 comes from performing $d/dt \cdot dy/dt$ and using "algebra" the symbols.



- If $x = f(t)$, then the symbol $f''(t)$, is used for the second derivative. Sometimes just f'' is used if it's clear what the independent variable is.

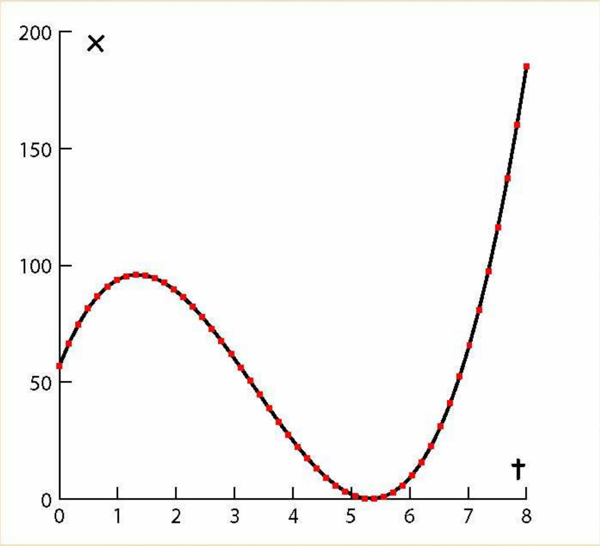
Example

- A math book moves in the x-direction with a displacement given by $x = 3t^3 - 30t^2 + 64t + 57$ meters, with t in seconds.

a) Find equations for its velocity and acceleration.

$v = dx/dt = 9t^2 - 60t + 64$ m/s

$a = dv/dt = d^2x/dt^2 = 18t - 60$ m/s^2



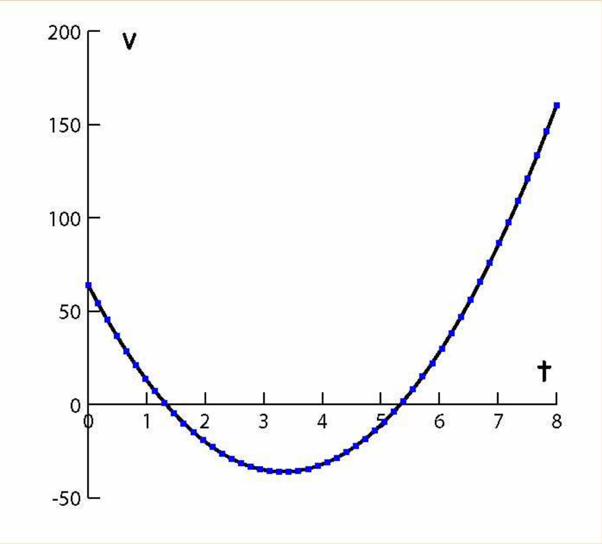
Example

b) At $t = 2$ sec, is the book's displacement increasing or decreasing? How fast?

$v = 9t^2 - 60t + 64$ m/s

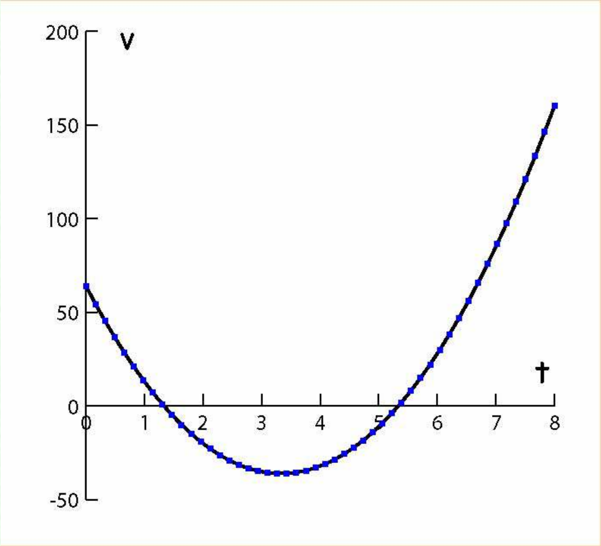
at $t=2$, $v = 9 \cdot 2^2 - 60 \cdot 2 + 64 = -20$ m/s

The book's position is decreasing at 20 m/s.



Example

c) At $t = 2$ sec, is the object speeding up or slowing down? At what rate?



$a = 18t - 60\ \ m/s^2$

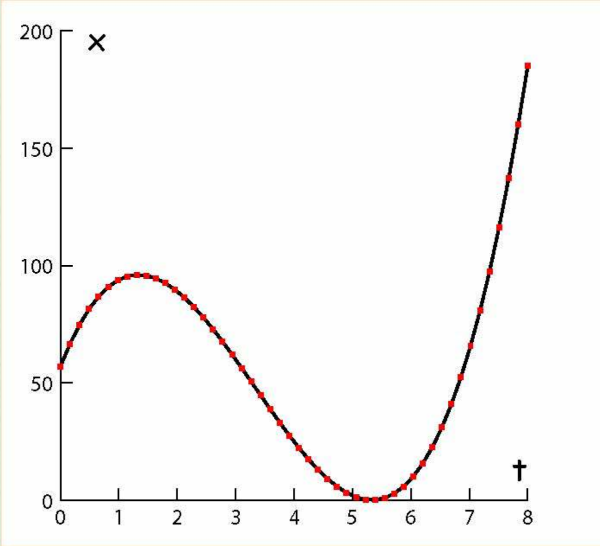
at $t = 2$ sec, $a = 18 \cdot 2 - 60 = -24\ m/s^2$

The book is speeding up in the negative direction.

Example

d) At what times in the interval [0,8] is x at a maximum?

The maximum (or minimum) of a function occurs
at either an endpoint
or when the first derivative is zero.



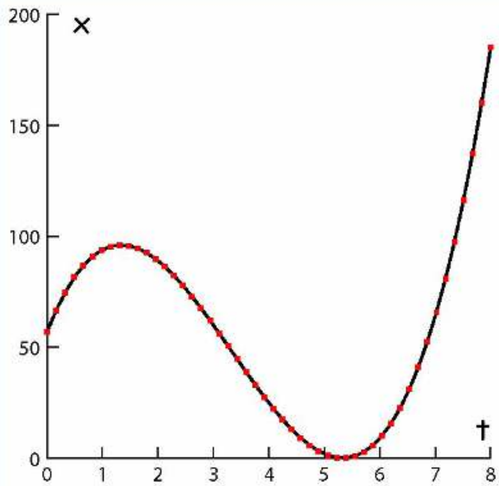
You can see from the graph that
one maximum occurs at the endpoint
when t = 8 sec.

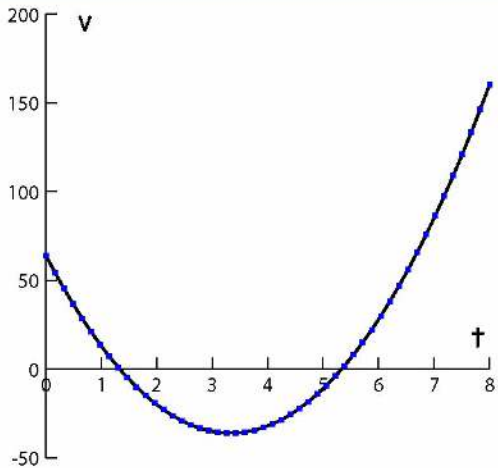
Example

To find the other minimum and/or maximum values, set the speed to zero:

$9t^2 - 60t + 64 = 0$ —> $t = 1.33$ and 5.33 sec, by the quadratic equation.

The plots show the $t = 1.33$ value is a local max while the $t = 5.33$ value is a min.



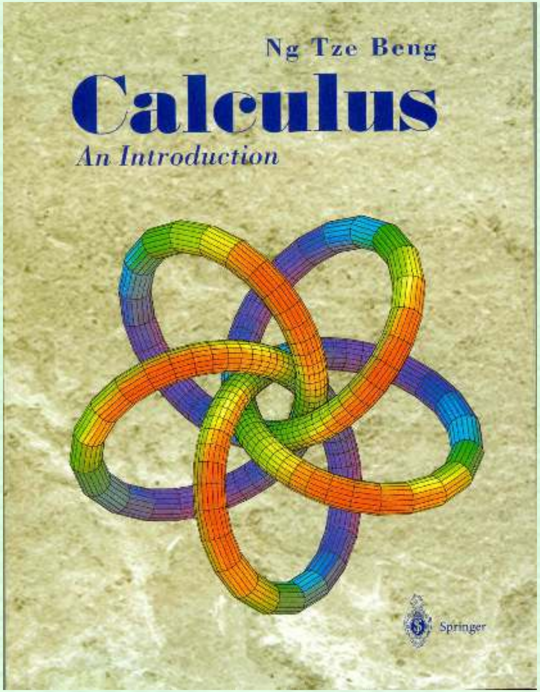


Example

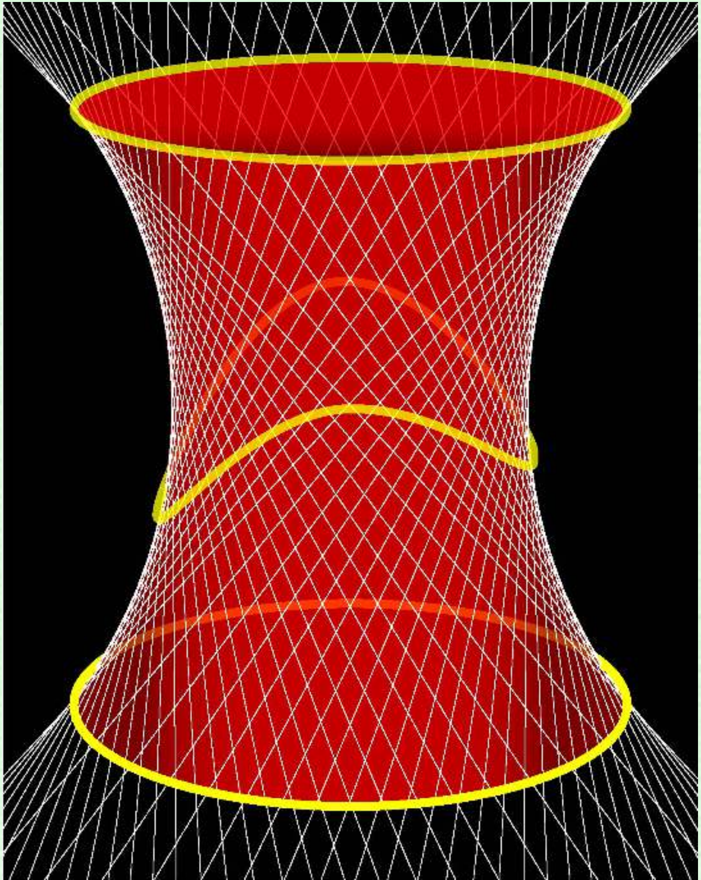
e) Is x ever negative in the [0,8] interval?

We found that a minimum occurs at t = 5.33 sec. Plugging this value of the time into the position equation gives a distance of 0.111 m.

Since x is still positive at the lowest point, it is never negative.

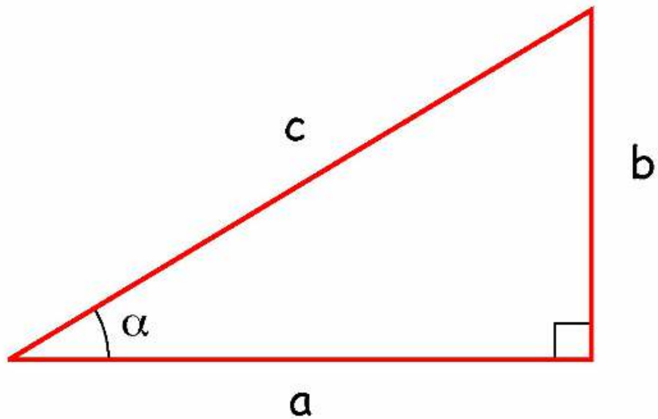


Interlude



Circular functions

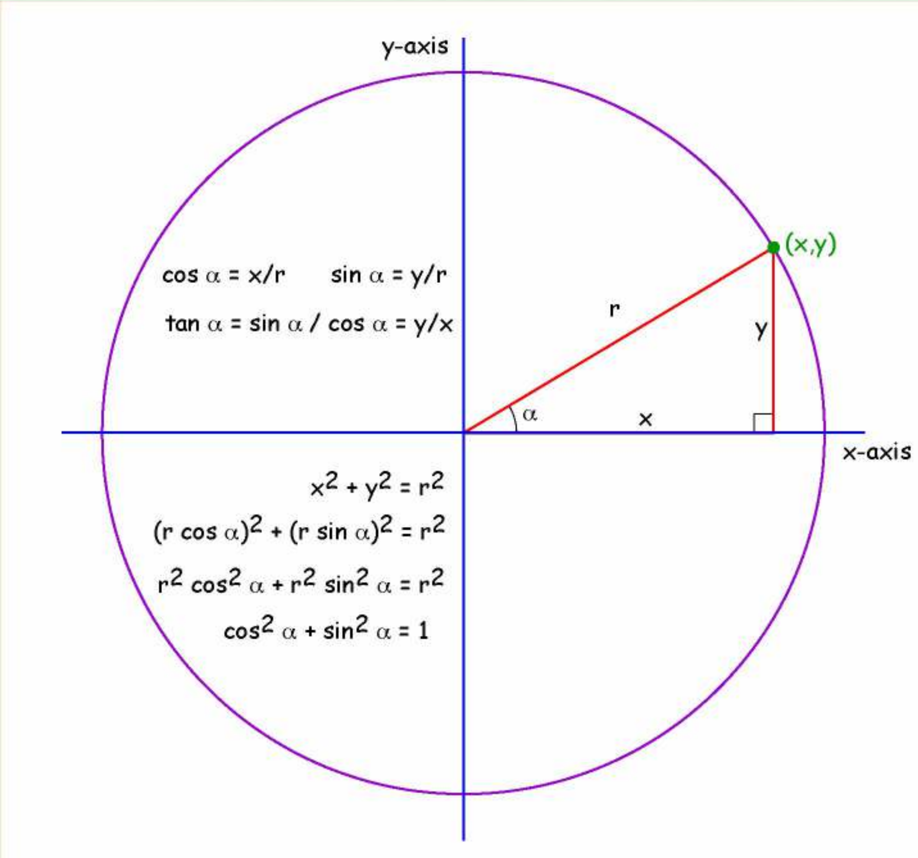
- The cosine, sine, and tangent are the three basic circular functions.



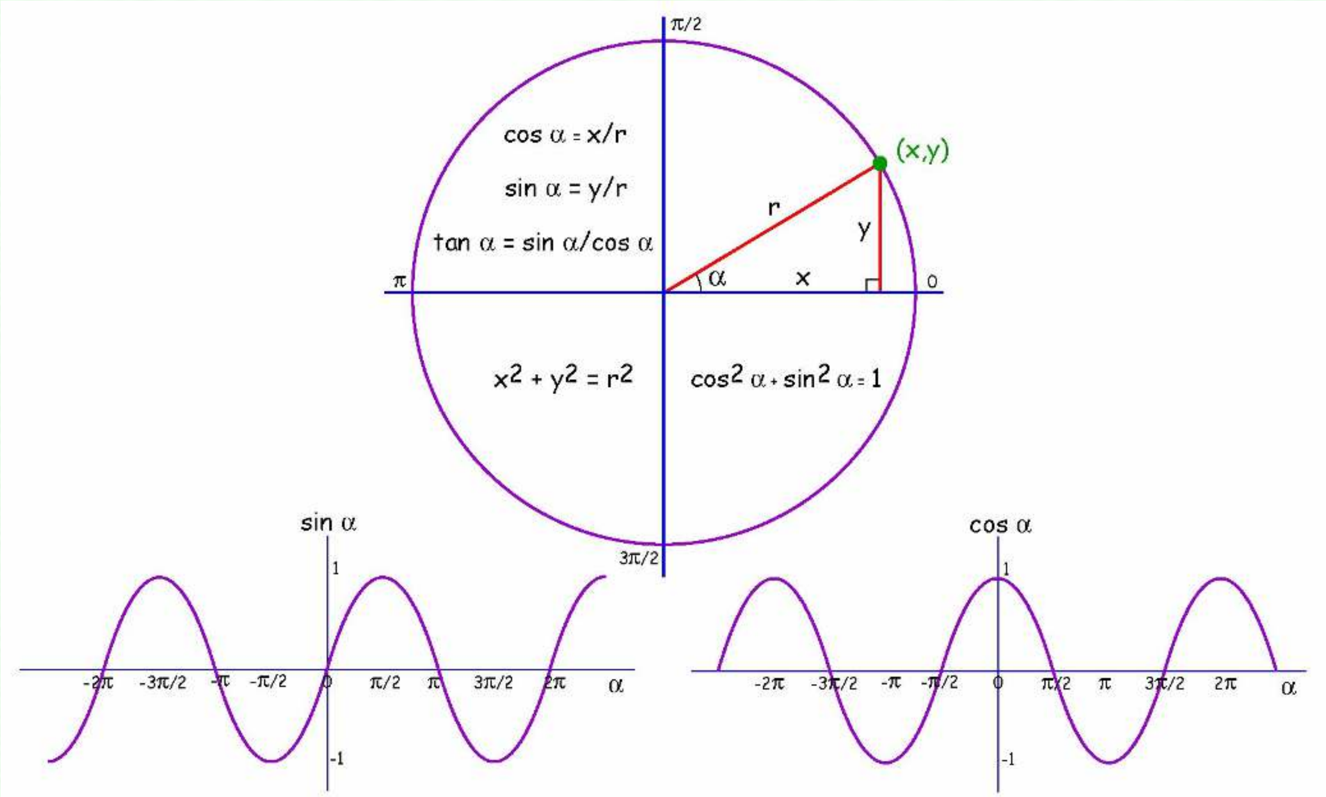
$\cos\alpha = a/c \qquad \sin\alpha = b/c$

$\tan\alpha = \sin\alpha / \cos\alpha = b/a$

Circular functions



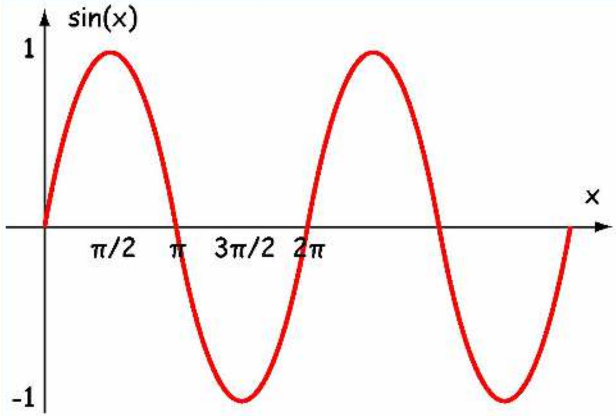
Circular functions

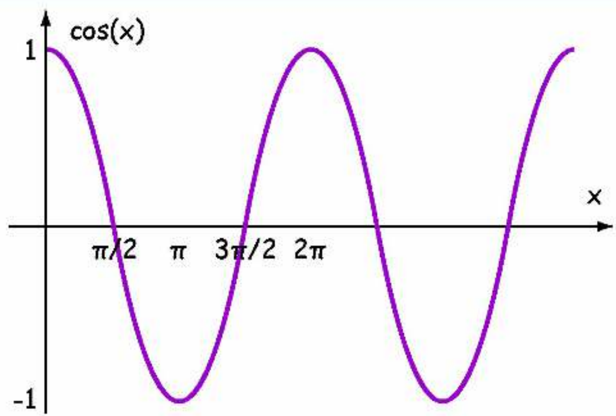


Sinusoids

- We've discovered how to write down a simple formula for the derivative of a linear combination of power functions, $d(y^n)/dt = n \cdot y^{n-1}$.

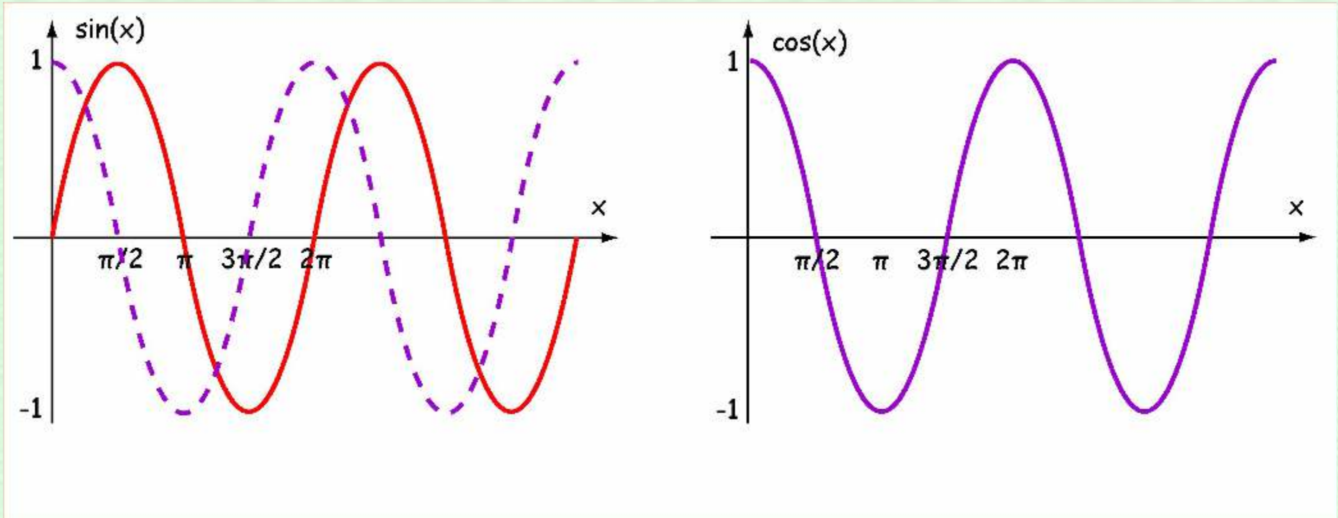
- Now we'll do this for the trigonometric functions sine and cosine.
  Any intuitive conjectures for what the derivative of sin x is?





Sinusoids

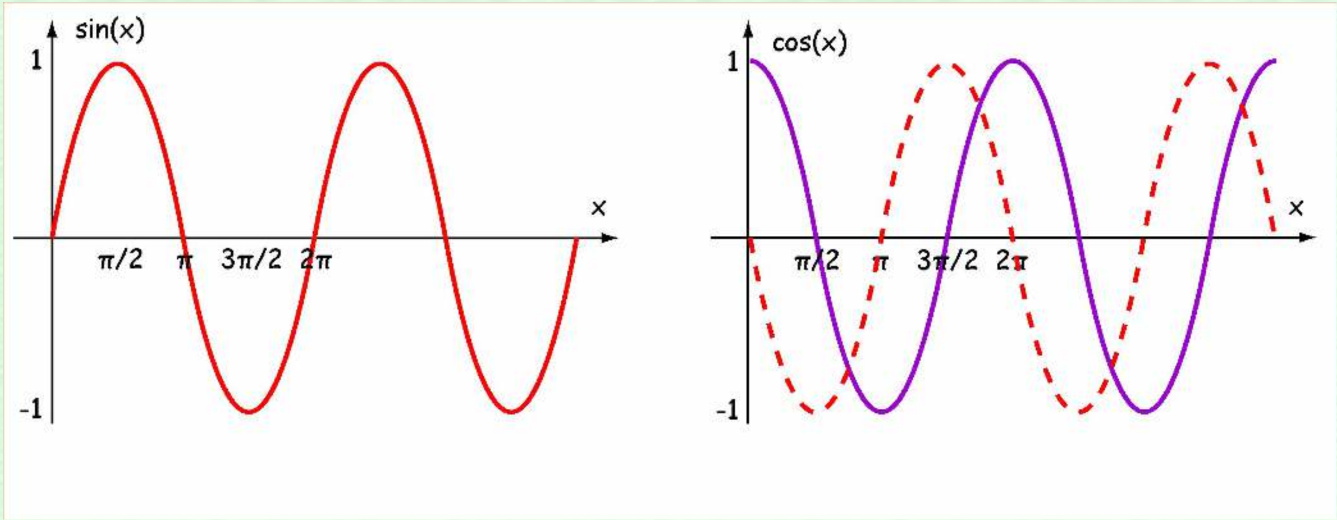
- Consider a graph of the numerical derivative of the sin x , shown as dashes.



- Do you have a guess for what the derivative of sin x is?

Sinusoids

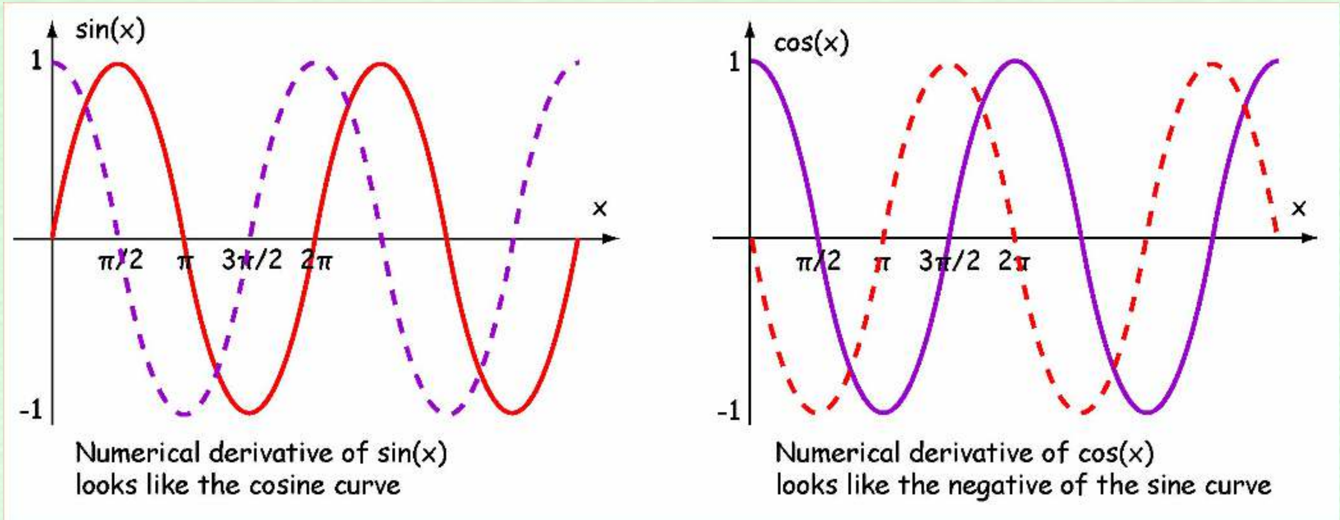
- Consider a graph of the numerical derivative of the cos x , shown as dashes:



- Do you have a guess for what the derivative of cos x is?

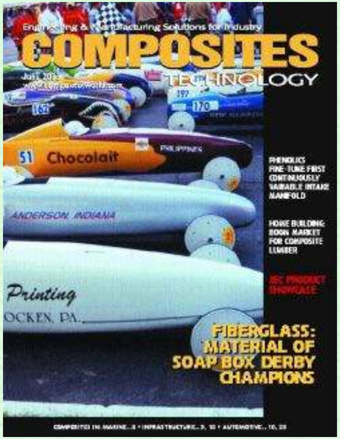
Sinusoids

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$



Composites

- Now that we know how to take the derivative of power functions x^n and the trigonometric functions sin x and cos x, we'll see if we can find the rule for what happens when we nest them.



Composites

- A composite function is two functions, one inside the other. It's easy to think of them as functions of functions.

$f(x) = \sin(x^5)$ or $g(x) = \cos^4 x$

- The function performed first is called the inside function - the 5th power in f or the cosine function in g.

- The function performed second is called the outside function - the sine in f or the 4th power in g.

Composites

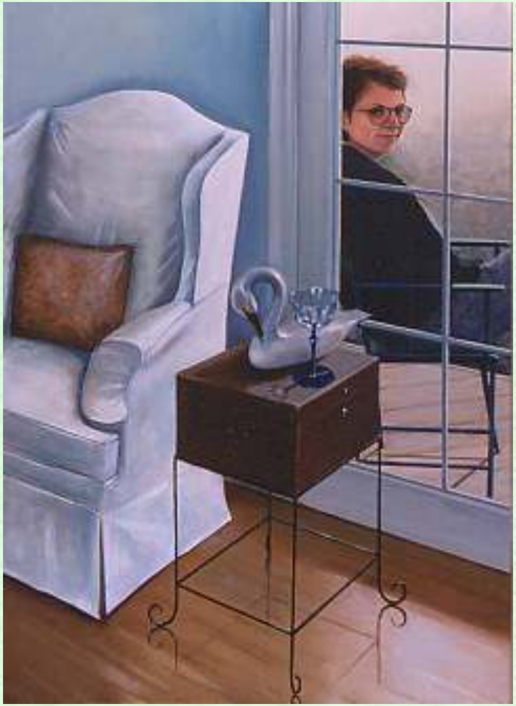
- Identify the inside and outside functions:

$f(x) = \sin(3x)$

$g(z) = 2^{\cos(z)}$

$p(t) = 1/\tan(t)$

$L(a) = \cos^3(\sin(a))$



Inside/Outside
2000, Anne Hancock
oil on panel

Chain rule

- The chain rule tells you how to differentiate a composite function:

$$\text{f(x) form} \quad : \text{ if } f = g(h(x)), \text{ then } f' = g'(h(x))\, h'(x)$$

$$\text{dy/dx form} : \text{ if } y = g(u) \text{ and } u = h(x), \text{ then } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

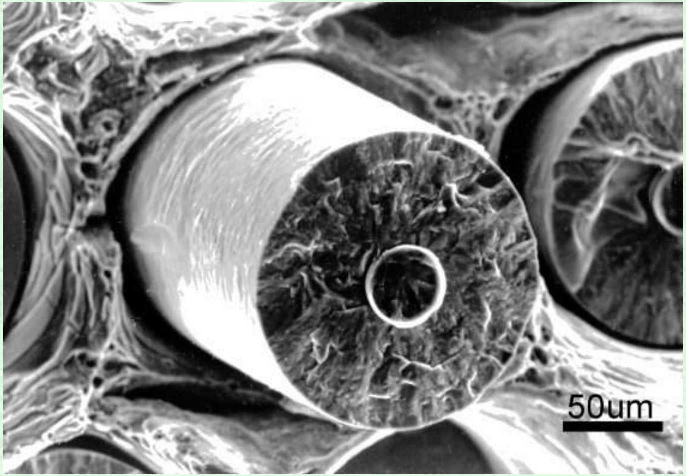
- To differentiate a composite, differentiate the outside function with respect to the inside function, and multiply by the derivative of the inside function.

Example

- Find the derivative of $f(x) = \sin(x^2)$.

Here the outside function is sine, and the inside function is x^2.

Applying the chain rule, $f' = g'h' = \cos(x^2) \cdot 2x$



Example

- Find the derivative of $y = (x^3 + 7)^2$.

Let $g(h) = u^2$ and $h(x) = x^3 + 7$. Applying the chain rule,

$$\frac{dy}{dx} = \frac{dg}{dh} \cdot \frac{dh}{dx}$$

$$= 2u \cdot 3x^2 = 2(x^3 + 7) \cdot 3x^2 = 6x^2(x^3 + 7)$$

Example

- Find the derivative of $p(t) = \cos^6 t$.

$p' =$ outside derivative • inside derivative $= 6\cos^5 t \cdot (-\sin t)$
$= -6\cos^5(t)\sin(t)$

Example

- Find the derivative of $z = (t^2 + 3t - 7)^{-5}$.

Let $g(h) = u^{-5}$ and $h(x) = t^2 + 3t - 7$. Applying the chain rule,

$$\frac{dy}{dx} = \frac{dg}{dh} \cdot \frac{dh}{dx}$$

$$= -5u^{-6} \cdot (2t + 3)$$

$$= -5(t^2 + 3t - 7)^{-6} \cdot (2t + 3)$$

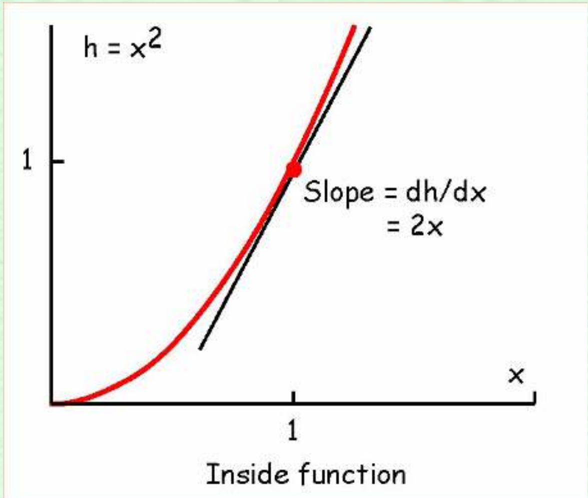
Chain rule

- To see why the chain rule works, realize that there are really three functions involved: the inside function, the outside function and the composite function. Each has its own derivative.

Chain rule

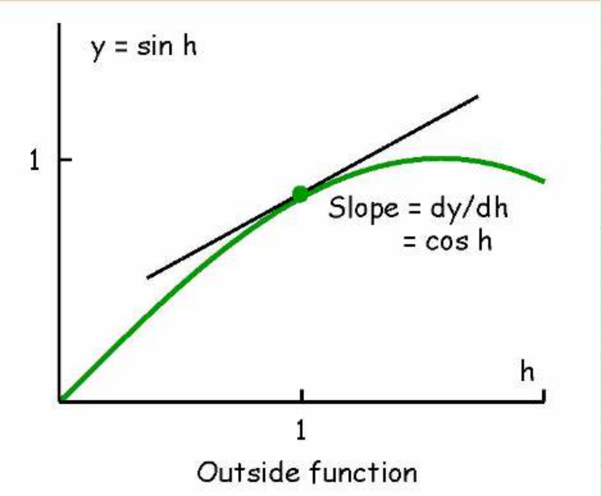
- Consider the composite function $y = \sin x^2$.
  The inside function is $h(x) = x^2$, and the outside function is $y = \sin h$.

- The derivative of h with respect to x, dh/dx, is 2x.
  It tells us the slope in the h-x plane.



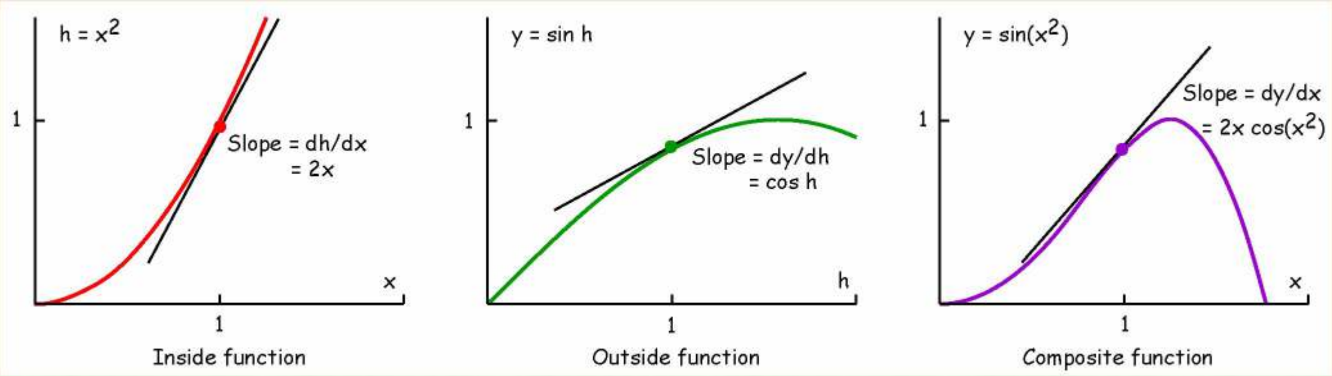
Chain rule

- Similarly, the derivative of y with respect to h, d (sin h) /dh is cos h. It tells the slope in the y-h plane.



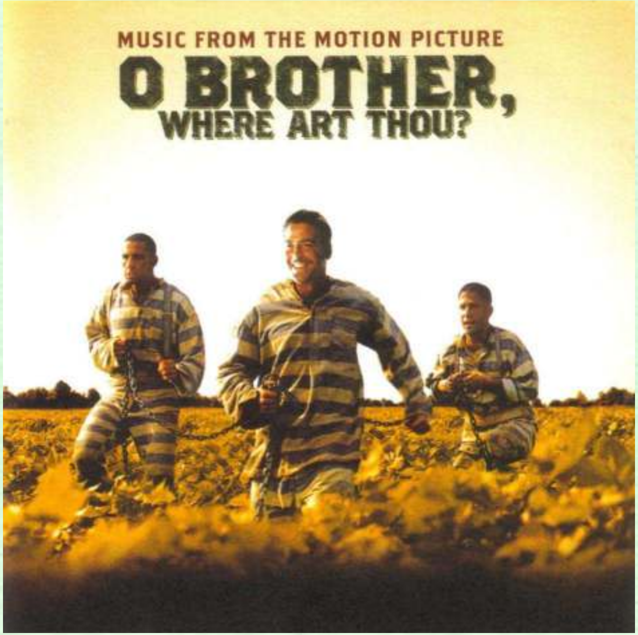
Chain rule

- We seek the derivative in the y-x plane, dy/dx.
  Being loose and treating the derivatives as fractions,
  we have dy/dx = dy/dh • dh/dx. The dh terms in some loose sense cancel.



Chain rule

- You can apply the chain rule to as many nested functions as you have. If the expression was three functions deep, $f(x) = a(b(c(x)))$, then the derivative of $f(x)$ with respect to x is $f' = a' \cdot b' \cdot c'$.



Example

- Find the derivative of $L(a) = \cos^3 [\sin (a^{-2} + 8a)]$

Applying the chain rule,

$$L' = 3 \cos^2[\sin(a^{-2} + 8a)] \cdot (-\sin[\sin(a^{-2} + 8a)]) \cdot \cos(a^{-2} + 8a) \cdot (-2a^{-3} + 8)$$

$$= -3 \cos^2(\sin(a^{-2} + 8a)) \cdot \sin(\sin(a^{-2} + 8a) \cdot \cos(a^{-2} + 8a) \cdot (8 - 2a^{-3})$$

- Imagine how ugly finding this derivative from the definition would be!

Playtime

- During our in-class problem solving session today you'll compute some polynomial and trigonometric derivatives, and explore the subtleties of the chain rule.