Research is what I am doing when
I don't know what I'm doing.

Wernher von Braun

School of the Art Institute of Chicago

Calculus

Frank Timmes

ftimmes@artic.edu

flash.uchicago.edu/~fxt/class_pages/class_calc.shtml

Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- library.thinkquest.org/10030/calcucon.htm
- ww.math.com/tables/derivatives/more/trig.htm
- www.ping.be/~ping1339/gonio.htm
- www-groups.dcs.standrews.ac.uk/~history/HistTopics/Trigonometric_functions.html

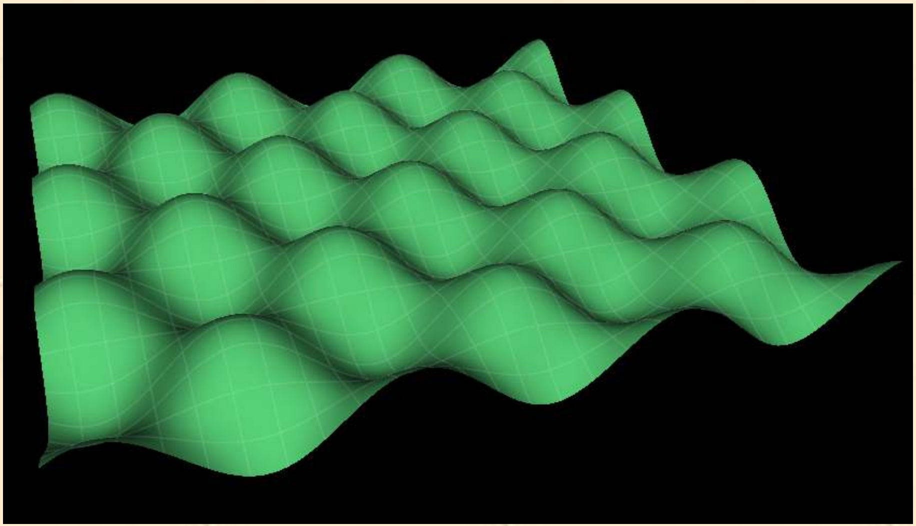
Class #8

- Sine and cosine derivatives
- Antiderivatives
- Derivative of a product

Sinusoids

- We've found that the derivative of a sine is a cosine $d(\sin x)/dx = \cos x$, and the derivative of a cosine is a negative sine $d(\cos x)/dx = -\sin x$.

- Let's show where these two derivatives come from.



l'Hôpital's rule

- Consider a case when two functions (f and g) and their derivatives (f' and g') are continuous.

- Suppose also that $$\lim_{x \to c} f(x) = 0 \text{ and } \lim_{x \to c} g(x) = 0$$

- One might think the quotient $$\lim_{x \to c} \frac{f(x)}{g(x)} \Rightarrow \frac{0}{0}$$

would be troublesome. But consider the following manipulation …

l'Hôpital's rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)}$$ since $f(a) = 0$ and $g(a) = 0$

$$= \lim_{x \to c} \frac{\dfrac{f(x) - f(c)}{x - c}}{\dfrac{g(x) - g(c)}{x - c}}$$ algebra

$$= \frac{f'(a)}{g'(a)}$$ by definition of $f'(a)$ and $g'(a)$

$$= \frac{\lim\limits_{x \to c} f'(x)}{\lim\limits_{x \to c} g'(x)}$$ f' and g' are continuous

$$= \lim_{x \to c} \frac{f'(x)}{g'(x)}$$ limit of a quotient property

l'Hôpital's rule

- What we've shown is called the zero-over-zero case of l'Hôpital's rule:

If f(x) and g(x) are differentiable over some interval

and $g'(x)$ is not zero in the interval

and $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$,

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

Example

- Find $\lim_{x \to 1} \dfrac{x^5 - 1}{x^3 - 1}$

In this case $f(x) = x^5 - 1$ and $g(x) = x^3 - 1$.
All of the assumptions of l'Hôpital's rule are satisfied.
In particular, $f(x)$ and $g(x)$ go to zero as x approaches one.

$$\lim_{x \to 1} \frac{x^5 - 1}{x^3 - 1} = \lim_{x \to 1} \frac{(x^5 - 1)'}{(x^3 - 1)'}$$

$$= \lim_{x \to 1} \frac{5x^4}{3x^2} = \lim_{x \to 1} \frac{5}{3} x^2 = \frac{5}{3}$$

Example

- Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$

In this case f(x) = 1 - cos x and g(x) = x.
f(x) and g(x) are differentiable and go to zero as x approaches zero.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{(1 - \cos x)'}{(x)'} = \lim_{x \to 0} \frac{\sin x}{1} = 0$$

Example

- Find $$\lim_{x \to 0} \frac{\sin x}{x}$$

In this case $f(x) = \sin x$ and $g(x) = x$.
$f(x)$ and $g(x)$ are differentiable and go to zero as x approaches zero.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{(\sin x)'}{(x)'} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

Sinusoids

- Let f(x) = sin x. By definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

- Algebra with fractions is usually easier if the numerator and denominator have one term each. The following identity can be used to transform the numerator from a sum to a product.

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

Here A = x + h and B = x. Applying this to the expression above gives:

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2\cos\frac{1}{2}[(x+h)+x]\,\sin\frac{1}{2}[(x+h)-x]}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2\cos(x+\frac{h}{2})\,\sin\frac{h}{2}}{h}$$

$$f'(x) = \lim_{h \to 0} \cos(x+\tfrac{h}{2}) \cdot \lim_{h \to 0} \frac{2\sin\frac{h}{2}}{h}$$

$$f'(x) = \cos x \cdot \lim_{h \to 0} \frac{2\sin\frac{h}{2}}{h}$$

$$f'(x) = \cos x \cdot \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \qquad \text{limit is 1 by l'Hopital's rule}$$

$$f'(x) = \cos x \cdot (1) = \cos(x)$$

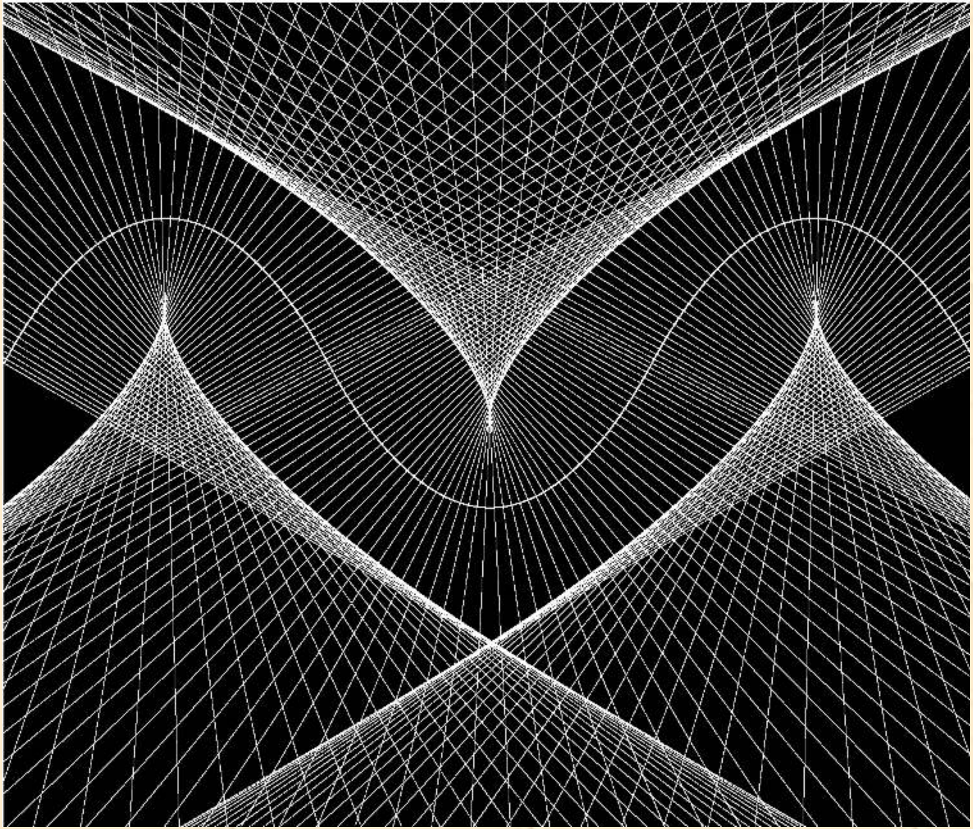
Sinusoids

- Using the relations between the sine and cosine functions, $\cos(\pi/2 - x) = \sin x$ and $\sin(\pi/2 - x) = \cos x$, makes finding the cosine derivative easy:

$$y = \cos(x) = \sin(\pi/2 - x)$$

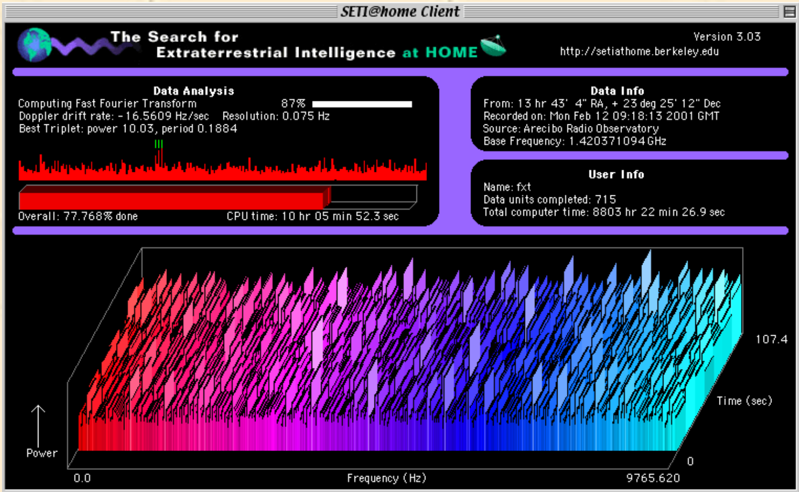
$$\begin{aligned} y' &= \cos(\pi/2 - x) \cdot (-1) \\ &= -\cos(\pi/2 - x) \\ &= -\sin(x) \end{aligned}$$

Interlude



Sinusoids

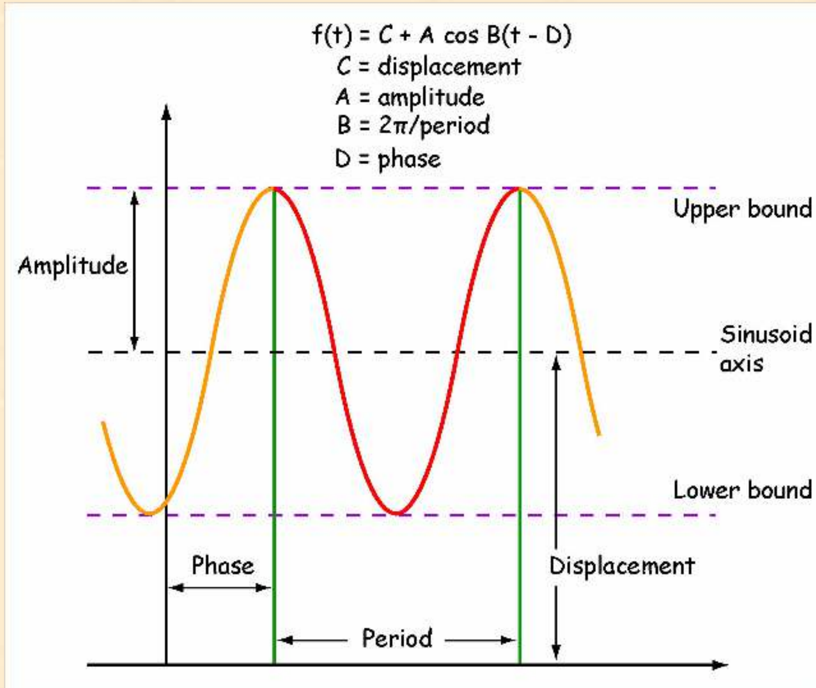
- Sine or cosine functions occur very frequently in the real world, particularly with periodic motion, image processing, and digital music.



- The general equation of a sinusoid is $f(t) = C + A \cos B(t - D)$, where the function can be either a sine or a cosine.

Sinusoids

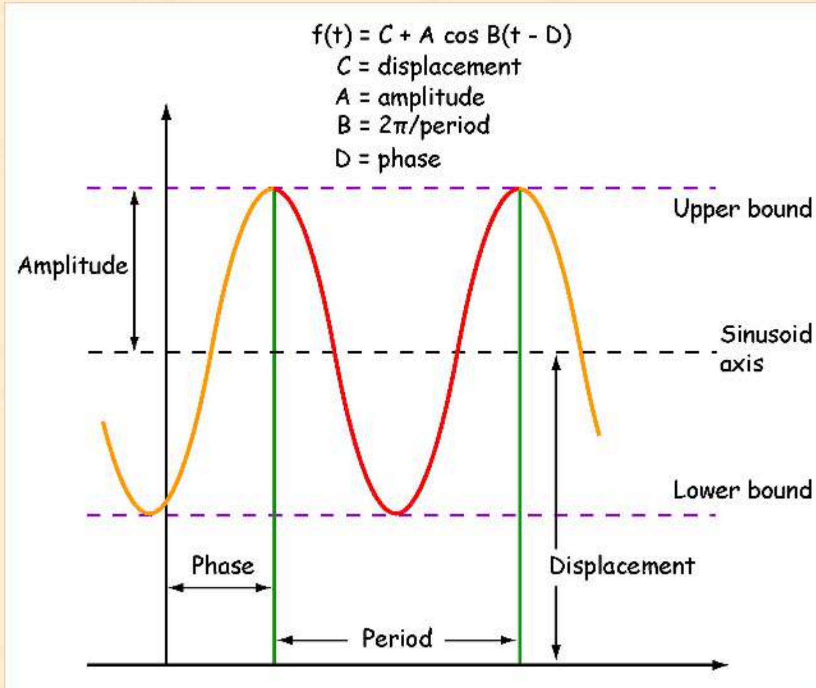
- The period is the number of units taken to complete one cycle.
  $B = 2\pi/\text{period}$



- The phase D is the coordinate of the beginning of a cycle, where the argument of the sinusoid is zero.

Sinusoids

- The amplitude A is the distance between the sinusoid axis and a high point.



- The displacement C is the distance from the x-axis to the sinusoid axis.

Sinusoids

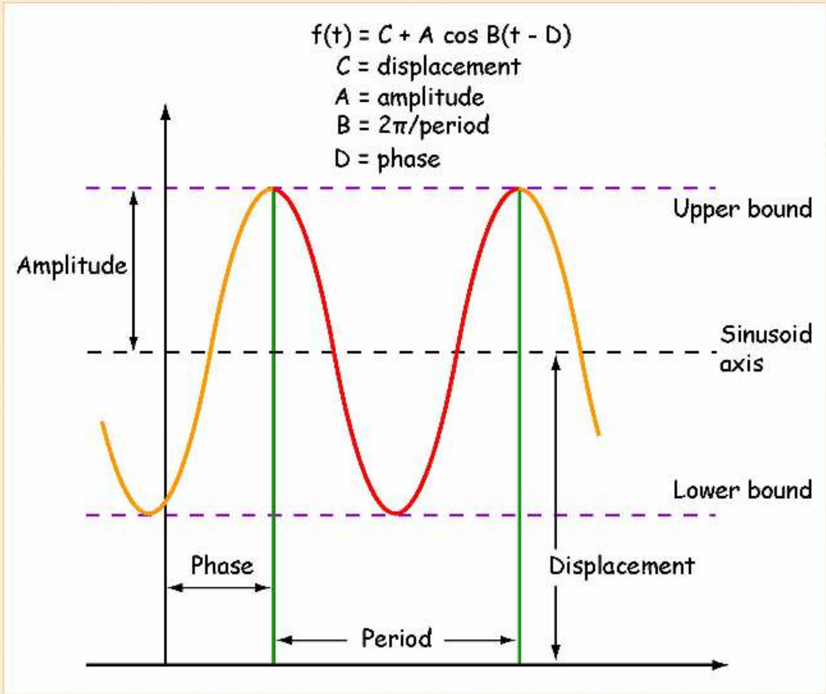
- Consider
  $f(x) = 5 + 3 \cos 2(x - 1)$

- What is the displacement?

  The amplitude?

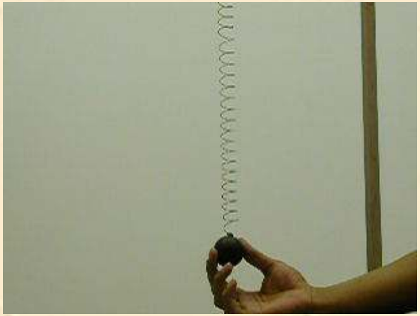
  Period?

  Phase?



Example

- A mass is bouncing up and down on a spring hanging from the ceiling.
  Its distance y from the ceiling varies sinusoidally with time t, making a complete cycle every 1.6 seconds.



- At $t = 0.4$ sec, y reaches its greatest value of 8 ft.
  The smallest y gets is 2 ft.

Example

a) Write an equation for y in terms of t.

The axis is halfway between the upper and lower bounds, so $C = 0.5 \cdot (2 + 8) = 5$.

The amplitude is from the axis to the upper bound, so $A = 8 - 5 = 3$.

The period is given, so $B = 2\pi/1.6 = 1.25\pi$.

A high point occurs at 0.4, so $D = 0.4$

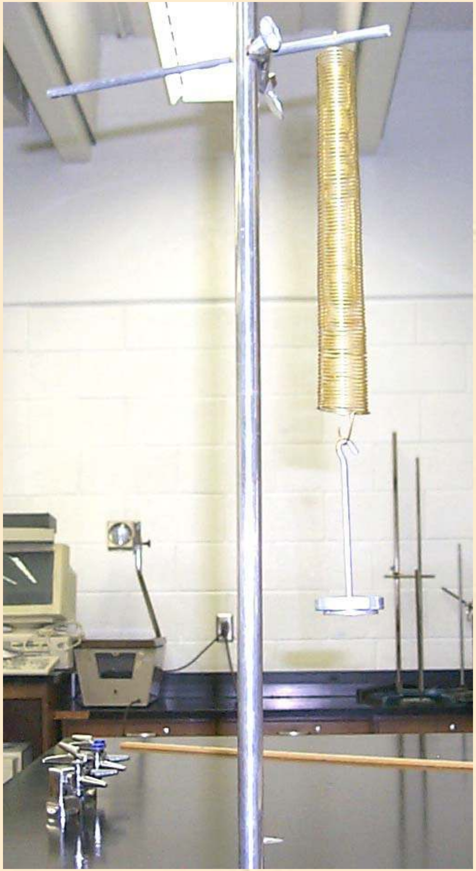
Thus, $y = 5 + 3 \cos 1.25\pi (t - 0.4)$ ft.

Example

b) Write an equation for the derivative y′.

$$y = 5 + 3\cos 1.25\pi\,(t - 0.4)\ \text{ft}$$

$$\begin{aligned} y' &= -3\sin 1.25\pi\,(t-0.4) \cdot 1.25\pi \\ &= -3.75\pi \sin 1.25\pi\,(t - 0.4)\ \ \text{ft/s} \end{aligned}$$



Example

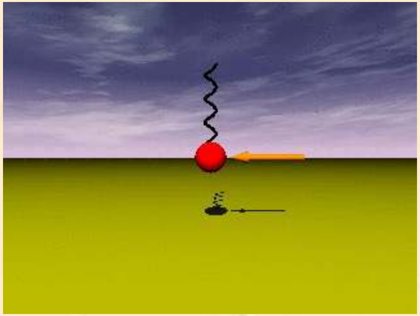
c) How fast is the mass moving at $t = 1$ s? $t = 1.5$ s? $t = 2.7$ s?
At $t = 2.7$ s, is the mass moving up or down?

Plugging these t values into the equation for y' gives:

$y'(t=1) = -8.3$ ft/sec

$y'(t=1.5) = 10.9$ ft/s

$y'(t=2.7) = -4.5$ ft/s



At $t = 2.7$ the mass is going up, but y' is negative so the distance between the mass and ceiling is getting smaller.

Example

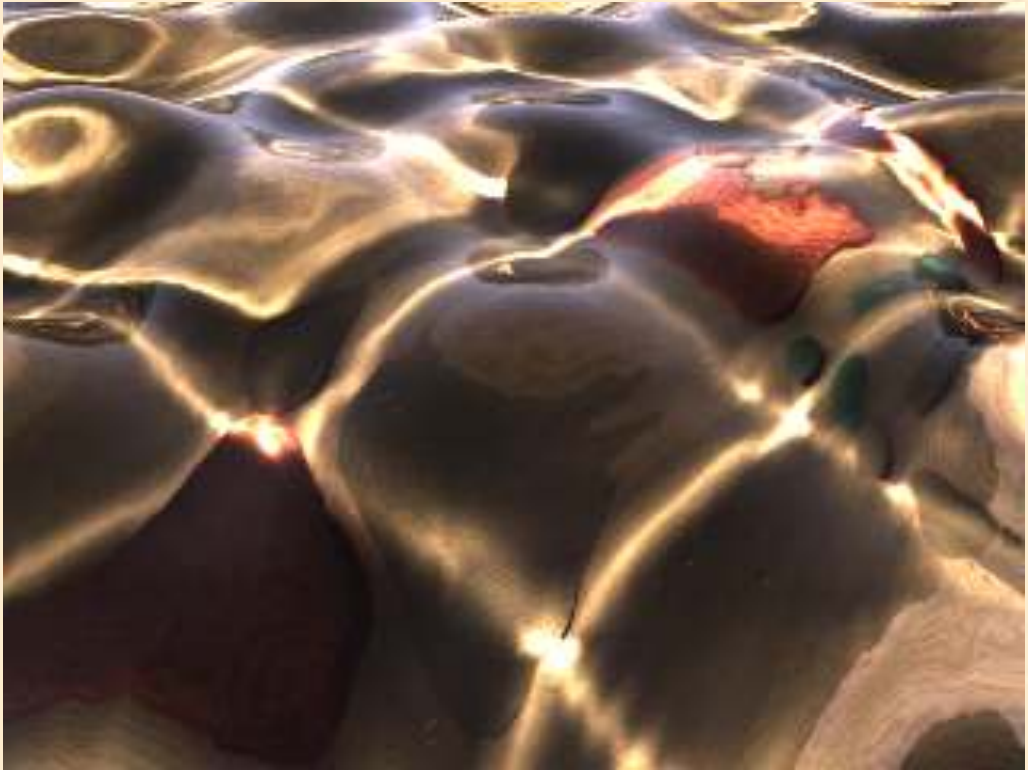
d) What is the maximum speed of the mass?
Where does the mass move this fast?

$y = 5 + 3 \cos 1.25\pi\,(t - 0.4)$ ft

$y' = -3 \sin 1.25\pi\,(t - 0.4) \cdot 1.25\pi = -3.75\pi \sin 1.25\pi\,(t - 0.4)$ ft/s

The fastest the mass moves is 3.75π ft/s, about 11.8 ft/s, which equals the amplitude of y'. At this point the mass is halfway between its high and low points.

Interlude



Antiderivatives

- If an object is falling freely under the action of gravity, it speeds up. You probably know that the speed near the surface of the earth is given by $v(t) = dy/dt = -9.8 \cdot t$ m/s.

- From this equation is it possible to find an equation for the position y?

Antiderivatives

- We have to work backwards from the process of taking the derivative, "What could we differentiate to get -9.8•t for the answer?"

- It we differentiate $y = t^2$, we get $dy/dt = 2 \cdot t$.
  The variable is correct, but its coefficient isn't -9.8.

- "What could we multiply 2 by to get -9.8?"
  The answer is -9.8/2, or -4.9.

- Thus, the function is $y = -4.9\ t^2$

Antiderivatives

- But, there are other functions we could differentiate to get -9.8t :

$y = -4.9\ t^2 + 3.7$ $\qquad$ $y = -4.9\ t^2 - 1776$ $\qquad$ $y = -4.9\ t^2 + \pi$

- The equation $y = -4.9t^2 + \text{constant}$ is called the general equation.
  Each particular constant that appears above is called a particular equation.

- An antiderivative is also called an indefinite integral, which we'll soon learn about.

Example

- If $f'(x) = x^7$, find the general equation for the antiderivative.

    - The exponent is 7, so the function differentiated must have had an exponent of 8.

    - The derivative of x^8 is $8x^7$, which is 8 times too big.
      So, the function differentiated must have been $1/8\ x^8$.

    - The function differentiated could also have had some constant added to it. The constant does not show up in the derivative because the derivative of a constant is zero.

    - Thus, the general equation would be $f(x) = 1/8\ x^8 + C$.

Antiderivatives

- The particular equation of an antiderivative can be found if we know the values of the function at one point.

- The coordinates of this point are called the initial conditions (one usually knows the values at the starting point).

- The initial conditions give us the information needed to find the particular value of the constant C.

Initial Conditions
1990 Paul Tzanetopoulos

Examples

- Galileo drops a cannonball from a leaning tower.
  Two seconds later the cannonball is at $y = 35$ m above the ground.

- If the velocity is given by
  $dy/dt = -9.8 \cdot t$,
  how tall is the leaning tower?

- When does the cannonball hit the ground?

Examples

$$dy/dt = -9.8 \cdot t \quad \longrightarrow \quad y = -4.9\, t^2 + C$$

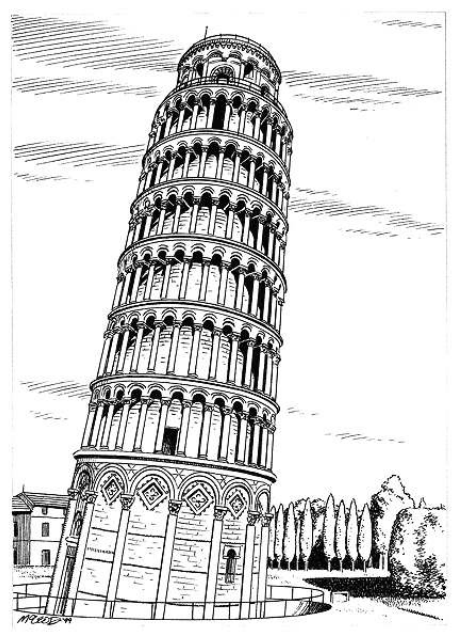
Substitute in the initial conditions ($y = 35$m at $t = 2$ s) to find the constant:

$$35 = -4.9 \cdot 2^2 + C \quad \longrightarrow \quad C = 54.6 \quad \longrightarrow \quad y = -4.9t^2 + 54.6 \text{ meters}$$

Galilei's inclined plane

Examples

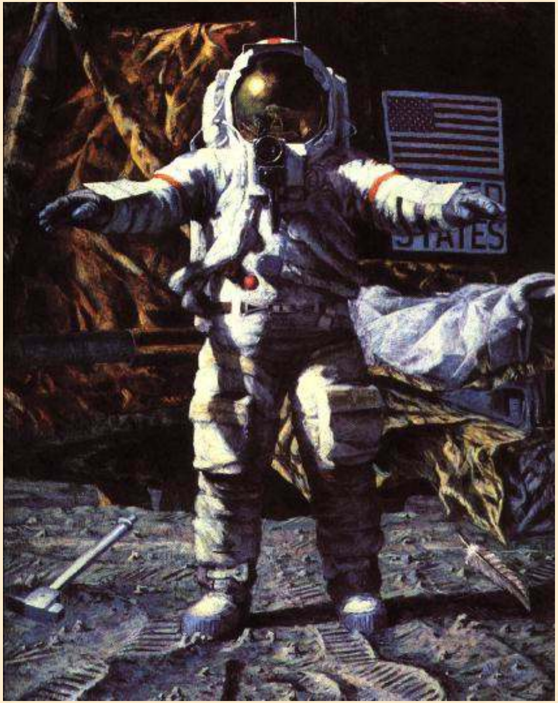
$$y = -4.9\ t^2 + 54.6 \quad m$$



The cannonball was dropped at $t = 0$.
At that time $y = 54.6$, so the tower is 54.6 m tall.

The cannonball hits the ground when $y = 0$.
$0 = -4.9\ t^2 + 54.6 \quad \longrightarrow \quad t = \mathrm{sqrt}(54.6/4.9) = 3.34$ sec

Interlude



The Hammer and the Feather
1990 Alan Bean, inspired by
Apollo 15 commander David R. Scott

Products

- We know that the derivative of a sum is the sum of the derivatives.

- Surprisingly, it turns out that the derivative of a product is not the product of the derivatives.

If $f = g \cdot h$, where g and h are differentiable functions, then $f' = g'h + gh'$

Derivative of the first times the second plus
the first times the derivative of the second.

Example

- If $y = x^4 \cos 6x$, find dy/dx.

$$dy/dx = 4x^3 \cos 6x + x^4(-\sin 6x)\cdot 6$$
$$= 4x^3 \cos 6x - 6x^4 \sin 6x$$
$$= 2x^3 (2 \cos 6x - 3x \sin 6x)$$

- As you write the product derivative, chant
  "derivative of first times second, plus first times derivative of second".
  Don't forget the chain rule!

Example

- If $y = (3x - 8)^7 (4x + 9)^5$, find y'.

$$y' = 7 \cdot (3x - 8)^6 \cdot 3 \cdot (4x + 9)^5 + (3x - 8)^7 \cdot 5 \cdot (4x + 9)^4 \cdot 4$$

$$= (3x - 8)^6 (4x + 9)^4 [21 (4x + 9) + 20 (3x - 8)]$$

$$= (3x - 8)^6 (4x + 9)^4 (144x + 29)$$

Products

$$f = g \cdot h$$

$$f' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f' = \lim_{\Delta x \to 0} \frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x)}{\Delta x}$$

Now form $g(x+\Delta x) = g + \Delta g$ and $h(x+\Delta x) = h + \Delta h$

$$f' = \lim_{\Delta x \to 0} \frac{(g+\Delta g)(h+\Delta h) - gh}{\Delta x}$$

$$f' = \lim_{\Delta x \to 0} \frac{(gh + g\Delta h + \Delta g h + \Delta g \Delta h) - gh}{\Delta x}$$

$$f' = \lim_{\Delta x \to 0} \frac{g\Delta h + \Delta g h + \Delta g \Delta h}{\Delta x}$$

$$f' = \lim_{\Delta x \to 0} \left(g\frac{\Delta h}{\Delta x} + h\frac{\Delta g}{\Delta x} + \Delta g \frac{\Delta h}{\Delta x} \right)$$

$$f' = g\frac{dh}{dx} + h\frac{dg}{dx} + 0\frac{dh}{dx}$$

$$f' = g'h + gh'$$

- Let's see why the product rule is true.

Playtime

- During your in-class problem solving session today you'll take the derivatives of some sinusoids and products, and you'll find some indefinite integrals.