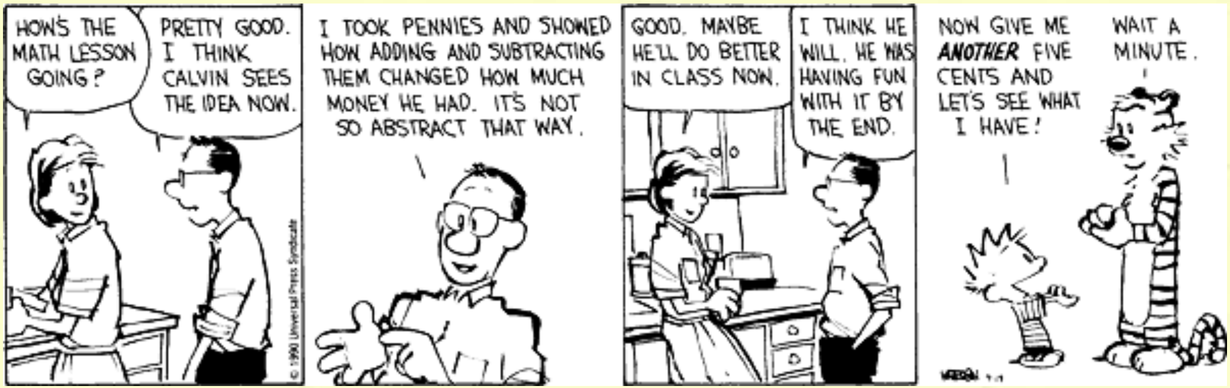
HOW'S THE MATH LESSON GOING?
PRETTY GOOD. I THINK CALVIN SEES THE IDEA NOW.
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I TOOK PENNIES AND SHOWED HOW ADDING AND SUBTRACTING THEM CHANGED HOW MUCH MONEY HE HAD. IT'S NOT SO ABSTRACT THAT WAY.
GOOD. MAYBE HE'LL DO BETTER IN CLASS NOW.
I THINK HE WILL. HE WAS HAVING FUN WITH IT BY THE END.
NOW GIVE ME **ANOTHER** FIVE CENTS AND LET'S SEE WHAT I HAVE!
WAIT A MINUTE.

School of the Art Institute of Chicago

Calculus

Frank Timmes

ftimmes@artic.edu

flash.uchicago.edu/~fxt/class_pages/class_calc.shtml

Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- web.mit.edu/wwmath/calculus/differentiation/quotients.html
- www.telusplanet.net/public/jwberger/mneumonx/math.html
- www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/trigderivdirectory/TrigDerivatives.html

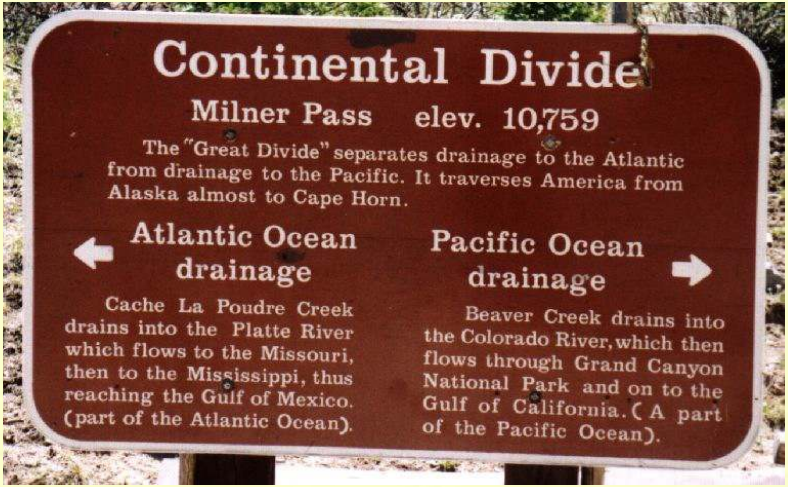
Class #9

- Quotient rule
- More trig functions
- Inverse functions

Divide

- Last time we explored the product rule for derivatives: $(fg)' = f'g + fg'$.

- What should we do when we divide two functions, f/g?



Divide

- Most texts, ours included, present the derivative of a quotient as an unrelated development and one more thing to store in the memory banks.

- I prefer to use the tools we already know and love: the product rule and the chain rule.

Divide

$$\left(\frac{f}{g}\right)' = \left(fg^{-1}\right)'$$

$$= f'g^{-1} + f(g^{-1})'$$

$$= f'g^{-1} + f(-g^{-2})g'$$

$$= \frac{f'}{g} - \frac{fg'}{g^2}$$

$$= \frac{f'g - fg'}{g^2}$$

Divide

- So, you can either remember the quotient rule formula

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

- Or simply write 1/g as g^{-1} and use the product and chain rules.

Example

- Differentiate $$y = \frac{\sin 5x}{8x - 3} = (\sin 5x)\cdot(8x-3)^{-1}$$

$$y' = (\cos 5x)\cdot 5\cdot(8x-3)^{-1} + (\sin 5x)\cdot(-1)\cdot(8x-3)^{-2}\cdot 8$$

$$= 5\,(\cos 5x)(8x-3)^{-1} - 8(\sin 5x)(8x-3)^{-2}$$

$$= \frac{5(\cos 5x)(8x-3) - 8\sin 5x}{(8x-3)^2}$$

Example

- Or we could use the formula: $$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$y = \frac{\sin 5x}{8x - 3}$$

$$y' = \frac{(\cos 5x) \cdot 5 \cdot (8x - 3) - (\sin 5x) \cdot 8}{(8x - 3)^2}$$

$$= \frac{5\,(\cos 5x)(8x - 3) - 8 \sin 5x}{(8x - 3)^2}$$

Example

- Differentiate $$y = \frac{(5x-2)^7}{(4x + 9)^3}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$y' = \frac{7(5x-2)^6 \cdot 5 \cdot (4x+9)^3 \; - \; (5x-2)^7 \cdot 3 \cdot (4x+9)^2 \cdot 4}{(4x+9)^6}$$

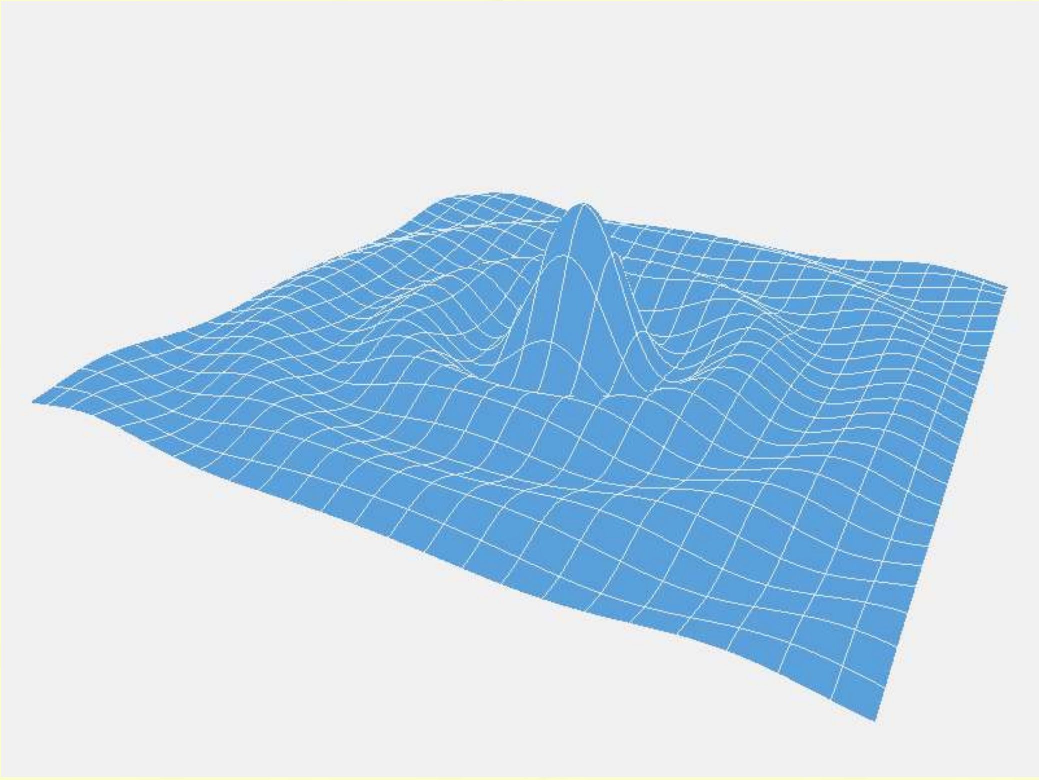
$$= \frac{35(5x-2)^6(4x+9)^3 \; - \; 12(5x-2)^7(4x+9)^2}{(4x+9)^6}$$

$$= \frac{35(5x-2)^6(4x+9) \; - \; 12(5x-2)^7}{(4x+9)^4}$$

$$= \frac{(5x-2)^6[35(4x+9) \; - \; 12(5x-2)]}{(4x+9)^4}$$

$$= \frac{(5x-2)^6(80x+339)}{(4x+9)^4}$$

Interlude

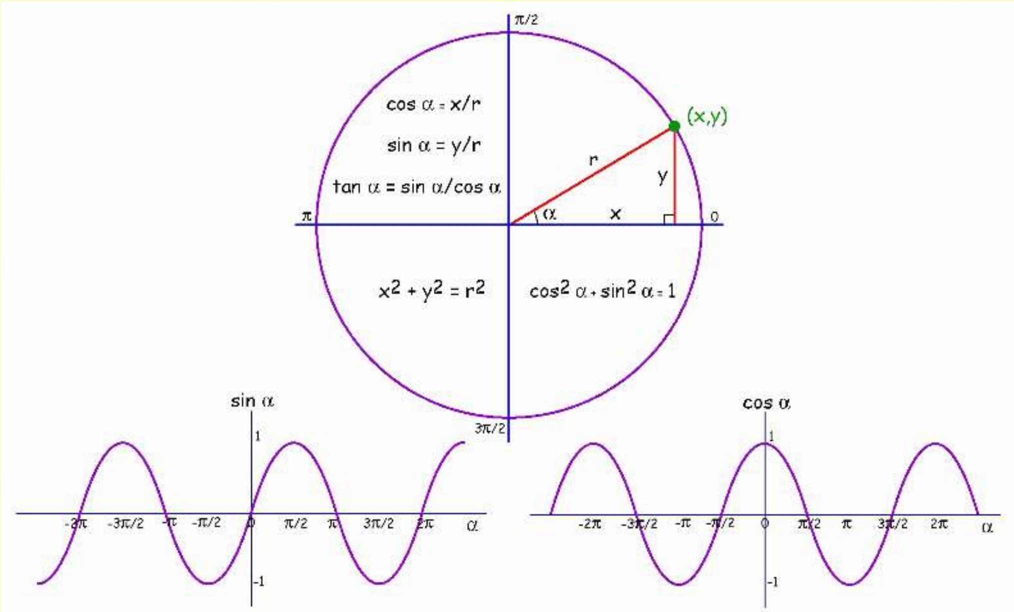


$$z = 6\,\frac{\sin(r - t)}{r}$$

Circular functions

- Recall the derivatives of the sine and cosine trigonometric functions,

$$d(\sin x)/dx = \cos x \qquad\qquad d(\cos x)/dx = -\sin x$$



Circular functions

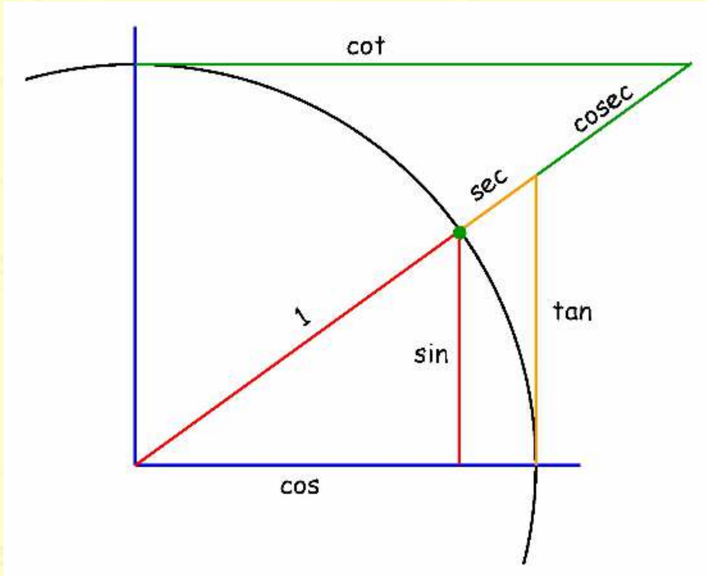
- With the tangent, cotangent, secant, and cosecant functions being defined in terms of the sine and cosine

tan x = sin x/cos x

cot x = cos x/sin x

sec x = 1/cos x

csc x = 1/sinx

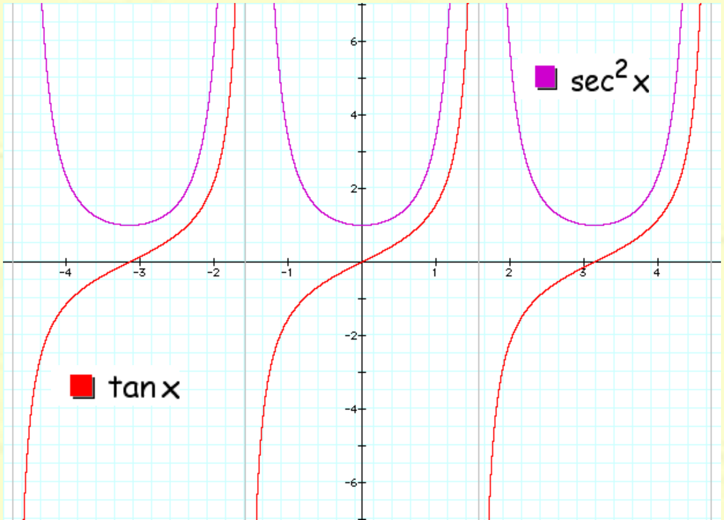


we can differentiate them since we know how to take derivatives of quotients.

Circular functions

$$y = \tan x = \frac{\sin x}{\cos x}$$

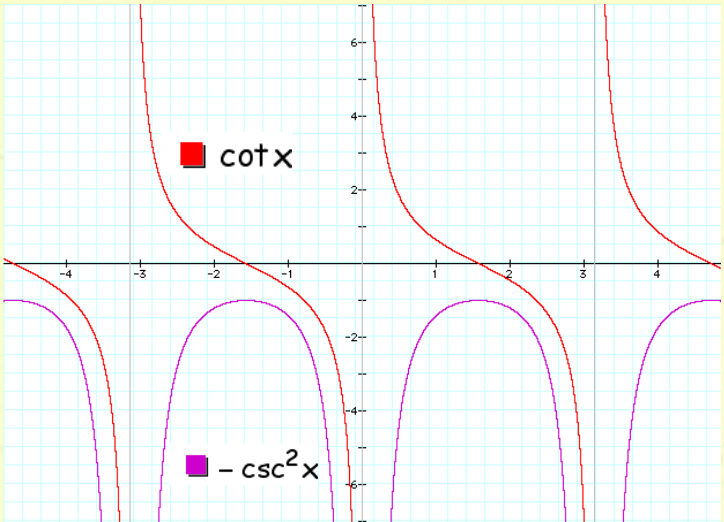
$$y' = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$



Circular functions

$$y = \cot x = \frac{\cos x}{\sin x}$$

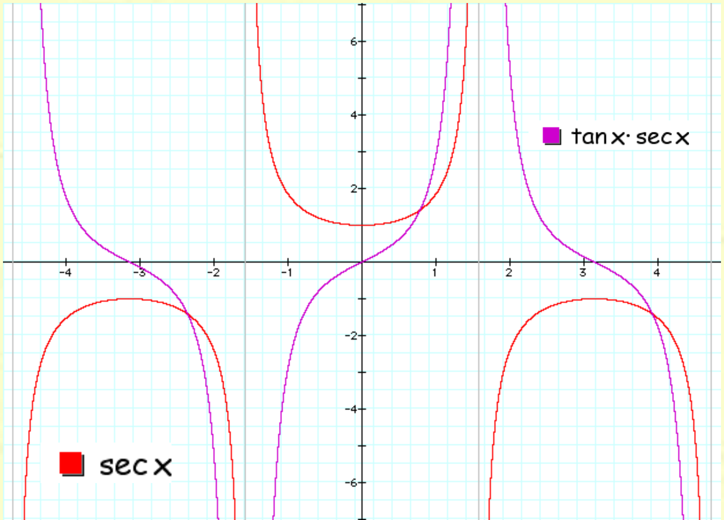
$$y' = \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\cos^2 x} = \frac{-1}{\cos^2 x} = -\csc x$$



Circular functions

$$y = \sec x = \frac{1}{\cos x}$$

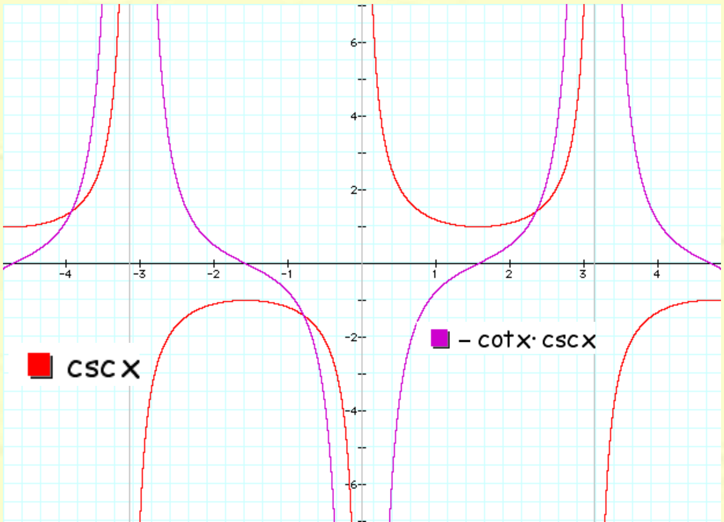
$$y' = \frac{(0)(\cos x) - (1)(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$



Circular functions

$$y = \csc x = \frac{1}{\sin x}$$

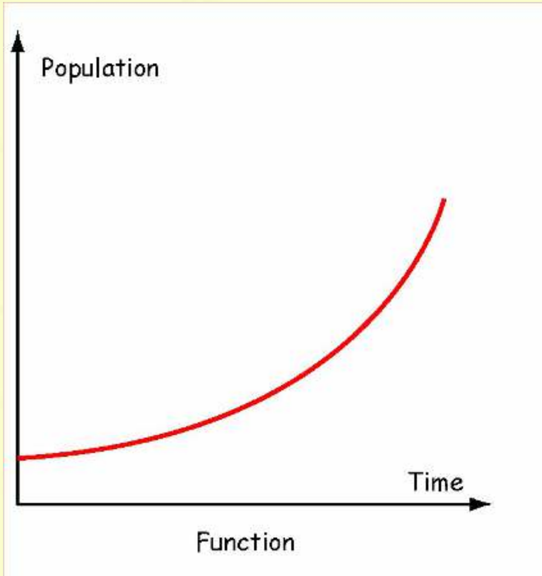
$$y' = \frac{(0)(\sin x) - (1)(\cos x)}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$



Interlude

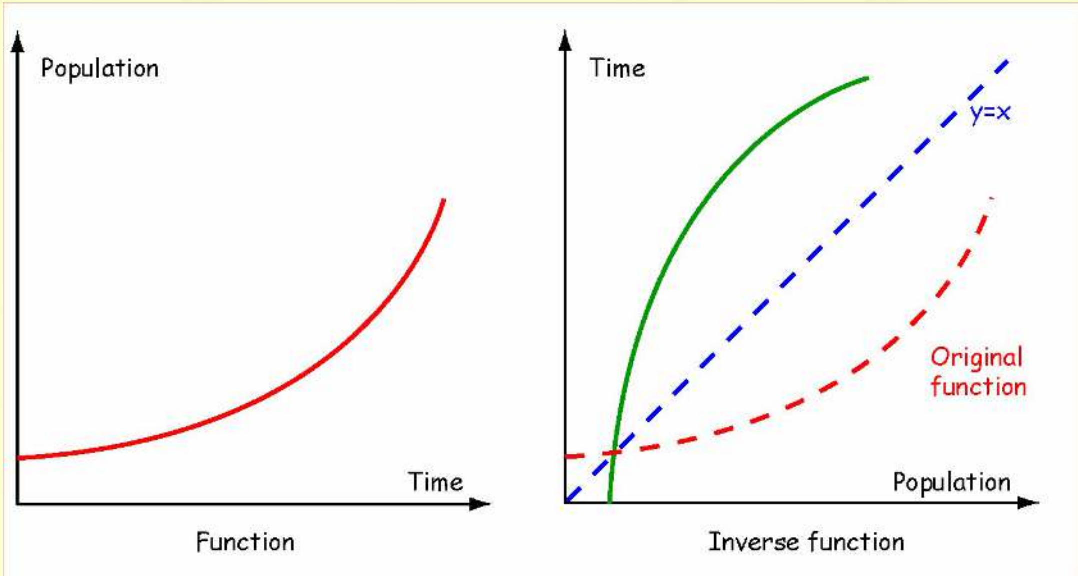
Inverse

- This plot shows how the population of this class may grow as a function of time.



Inverse

- If the dean was more interested in finding the time at which the population reached a certain value, it may be convenient to reverse the variables and write time as a function of population.



Inverse

- The relation you get by interchanging the two variables is called the inverse of the original function.

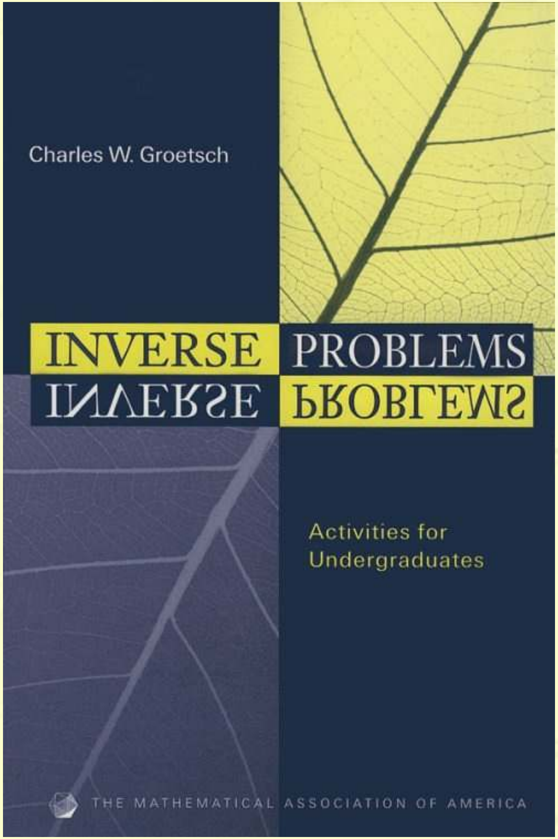
- The symbol f^{-1}, pronounced "f inverse" is used for the inverse function of f. Note that the -1 exponent is not the reciprocal of the function, $1/f = (f)^{-1}$.

Inverse

- For the linear function $y = 2x + 6$, interchanging variables gives $x = 2y + 6$. Solving for y gives $y = 0.5x - 3$.

- Here, $f(x) = 2x + 6$ and $f^{-1}(x) = 0.5x - 3$

- If f^{-1} turns out to be a function (one value of $f^{-1}(x)$ for any value of x), then the original function f is said to be invertible.



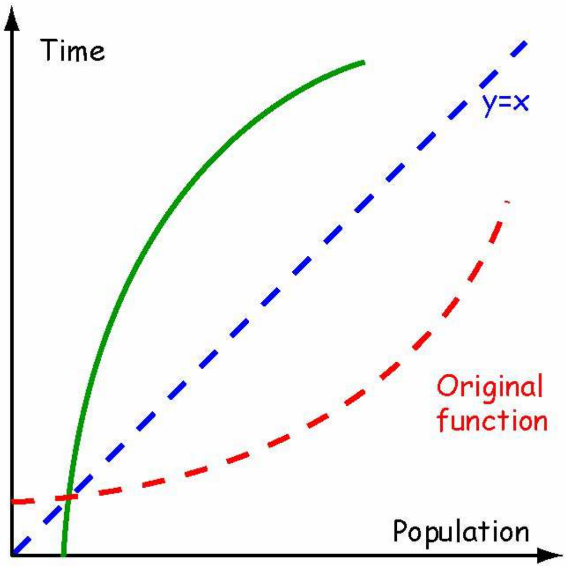
Inverse

- The inverse of a function undoes what the function did to x.
  That is, $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.
  For example, if $f(x) = x^2$, then $f^{-1}(x) = sqrt(x)$, and $sqrt(x^2) = x$.

The Man and the Vegetables (The Gardener)
Giuseppe Arcimboldo, circa 1560, oil on wood

Inverse

- Notice that if the same scales are used for the two axes, then plots of f and f^{-1} are mirror images with respect to the 45° line y = x.



Inverse function

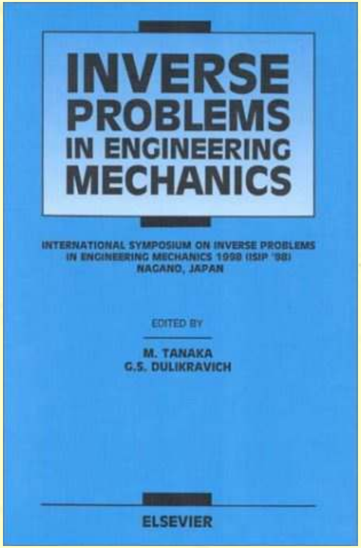
Inverse

- Definition:
  If $y = f(x)$, then the inverse of the function f has the equation $x = f(y)$

- Symbol:
  If $x = f(y)$, then $y = f^{-1}(x)$.

- Definition:
  If f^{-1} is a function, then f is said to be invertible.

- Property:
  If f is invertible, then $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

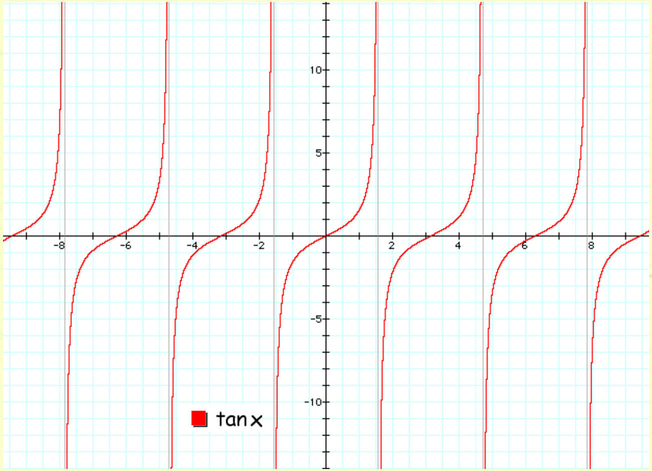


Trig inverses

- The inverses of the trigonometric functions follow from the definition:

  $y = \tan x$ function

  $x = \tan y$ inverse function

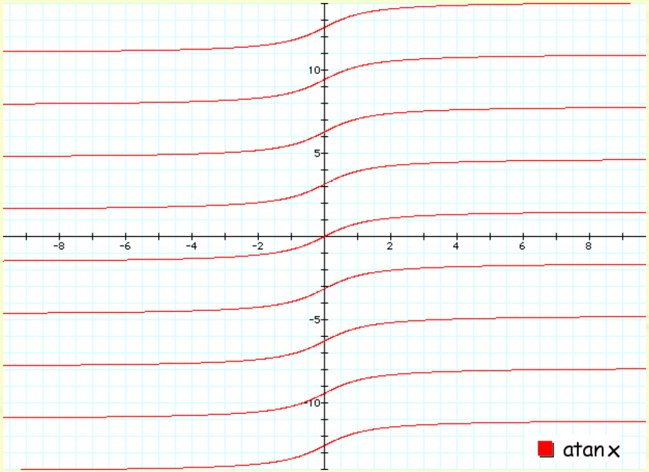


- In $x = \tan y$,
  y is called the inverse tangent of x,
  $y = \tan^{-1} x$.

- The symbol arctan ("the arc whose tangent is x") or atan is often used to help distinguish $\tan^{-1} x$ from $1/\tan x$.

Trig inverses

- Unfortunately, the inverse tangent is not a function.

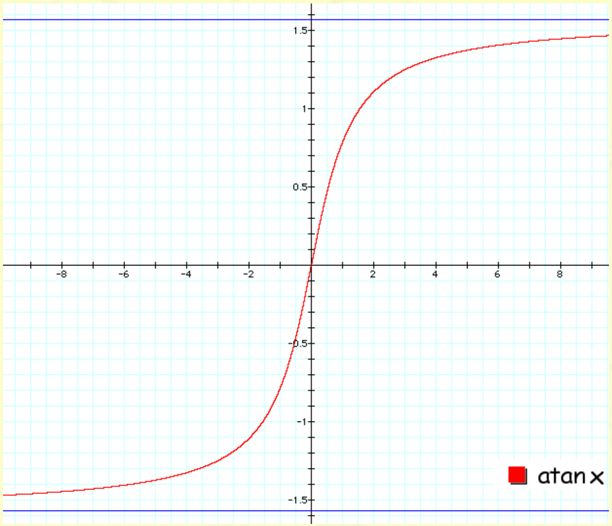


- There are many values of y for the same x value.

Trig inverses

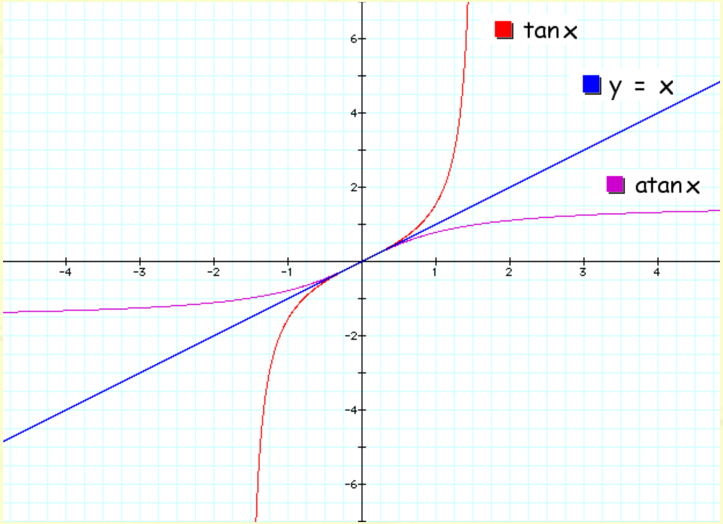
- To make a function that is the inverse of tan x,
  it is customary to restrict the range from $-\pi/2 < y < \pi/2$.

- This restriction includes only the branch through the origin, and is called the principal branch of the inverse tangent function.



Trig inverses

- Note that tan x and arctan x are symmetrical about the line y = x.

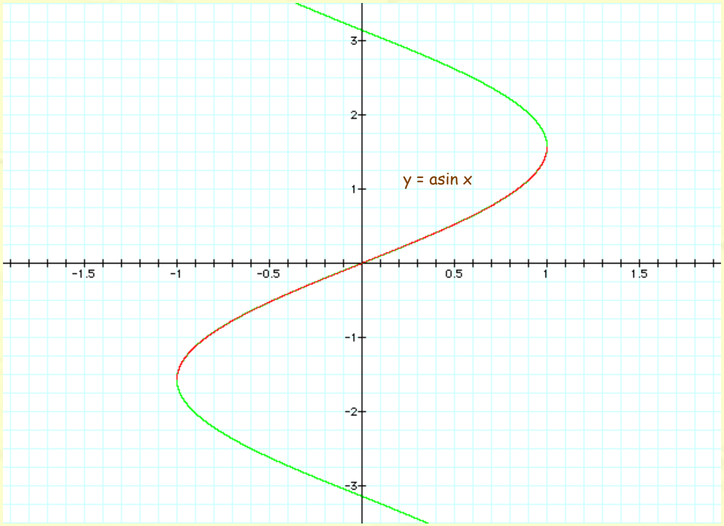


- Everything that is an x-feature on the tangent graph is a y-feature on the inverse tangent graph.

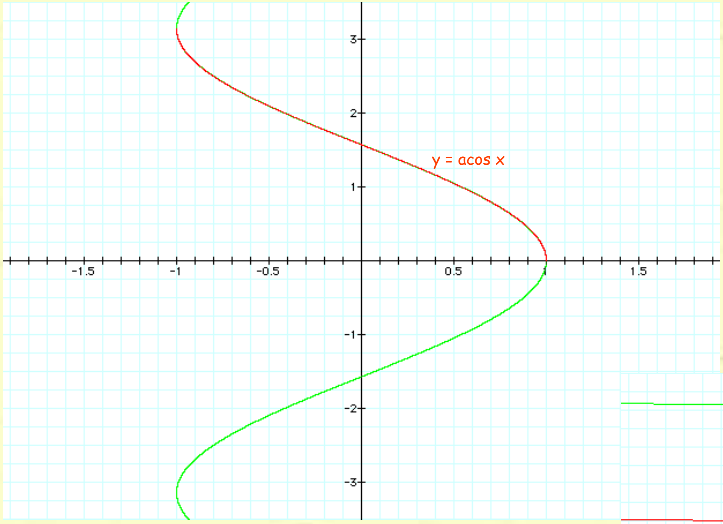
Trig inverses

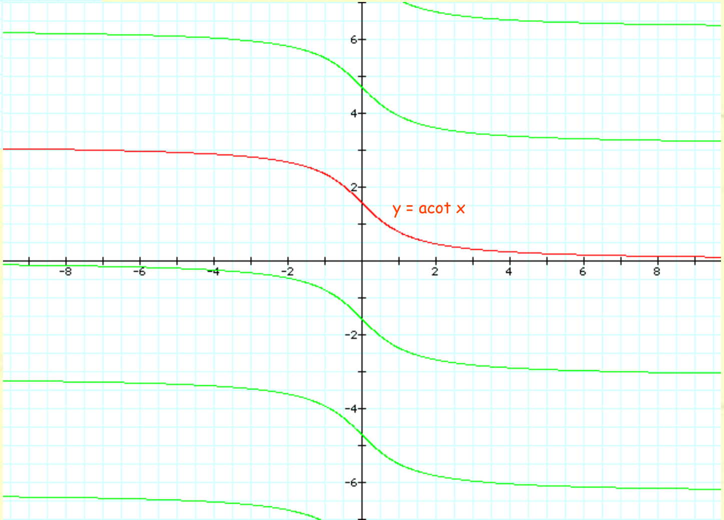
- The other five inverse trigonometric functions are defined the same way.

- For each, a principal branch near the origin, continuous if possible and positive if there is a choice between branches, is chosen in order to make the inverse relation a function.

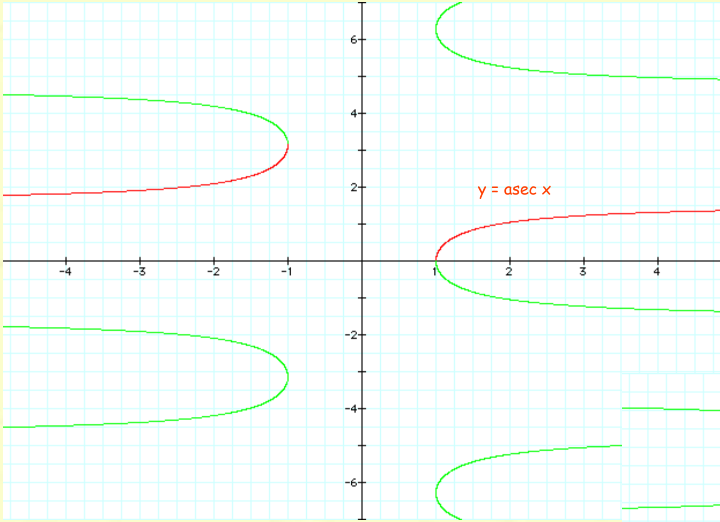


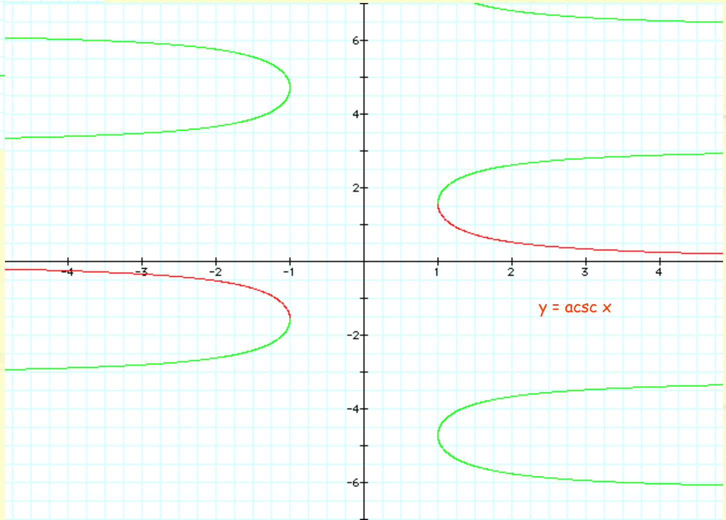
Trig inverses





Trig inverses





Interlude

"Self-portrait" and "Portrait of Rudolf II as Vertumnus"
Giuseppe Arcimboldo, circa 1560

Inverse derivatives

- Here is how to differentiate the inverse tangent function. The definition of the inverse function turns this new problem into the old problem of differentiating the tangent.

- Differentiate $y = \arctan x$ with respect to x.

Since $y = \arctan x$, this means $\tan y = x$.

The derivative of the tan is $\sec^2$. Because y depends on x, it is an inside function and we must apply the chain rule:

$$\sec^2 y \cdot y' = 1$$

$$y' = 1/\sec^2 y = \cos^2 y$$

Inverse derivatives

So we have $y = \arctan x$, $\tan y = x$, and $y' = \cos^2 y$.

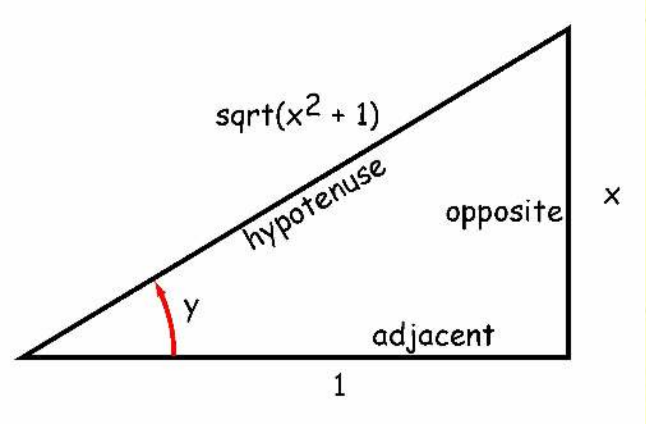
To find y' in terms of x, consider a triangle whose angle is y.
Since tangent = opposite/adjacent = $x = x/1$, we put x on the opposite leg and 1 on the adjacent leg.

This means the hypotenuse is $\mathrm{sqrt}(x^2+1)$. With the cosine equal to the adjacent/hypotenuse, we finally have

$y' = \cos^2 y$

$= 1/[\mathrm{sqrt}(x^2+1)]^2$

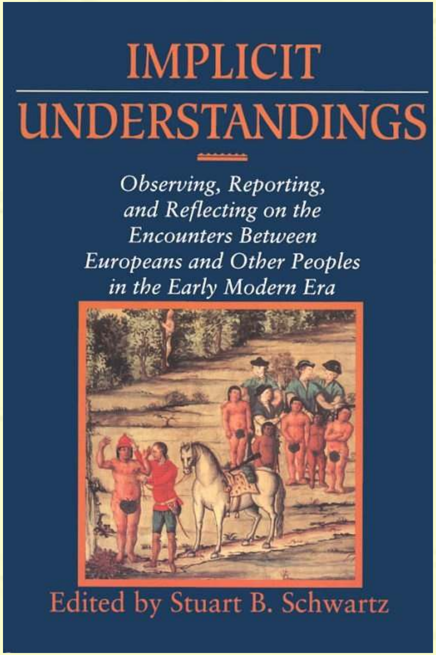
$= 1/(x^2+1)$



Inverse derivatives

- When we differentiated both sides of $\tan y = x$ with respect to x to form $\sec^2 y \cdot y' = 1$, we were applying the technique of implicit differentiation.

- We'll study this technique more extensively next time when we look at parametric functions.



Inverse derivatives

- As you've probably gathered by now, the derivatives of the inverse trigonometric functions are a bit involved. Here they are:

$$\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d(\arccos x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2} \qquad \frac{d(\text{arccot } x)}{dx} = -\frac{1}{1+x^2}$$

$$\frac{d(\text{arcsec } x)}{dx} = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d(\text{arccsc } x)}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}$$

Example

- Differentiate $y = \arccos 5x^7$

$$\text{Using } \frac{d(\arccos x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$y' = -\frac{1}{\sqrt{1-(5x^7)^2}} \cdot 35x^6$$

$$y' = -\frac{35x^6}{\sqrt{1-25x^{14}}}$$

Playtime

- During your in-class problem solving session today you'll take some derivatives of quotients, and know the trig functions better than you did when you walked in here!