The mathematician's best work is art, a high perfect art, as daring as the most secret dreams of imagination, clear and limpid. Mathematical genius and artistic genius touch one another.

Gosta Mittag-Leffler

School of the Art Institute of Chicago

# Calculus

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flash.uchicago.edu/~fxt/class_pages/class_calc.shtml

# Syllabus

| | | |
|---|---|---|
| 1 | Aug 29 | Pre-calculus |
| 2 | Sept 05 | Rates and areas |
| 3 | Sept 12 | Trapezoids and limits |
| 4 | Sept 19 | Limits and continuity |
| 5 | Sept 26 | Between zero and infinity |
| 6 | Oct 03 | Derivatives of polynomials |
| 7 | Oct 10 | Chain rule |
| 8 | Oct 17 | Product rule and integrals |
| 9 | Oct 24 | Quotent rule and inverses |
| 10 | Oct 31 | Parametrics and implicits |
| 11 | Nov 7 | Indefinite integrals |
| 12 | Nov 14 | Riemann sums |
| 13 | Dec 05 | Fundamental Theorem of Calculus |

# Sites of the Week

- archives.math.utk.edu/visual.calculus/0/parametric.6/
- web.mit.edu/wwmath/calculus/differentiation/implicit.html
- astronomy.swin.edu.au/~pbourke/surfaces/

# Class #10

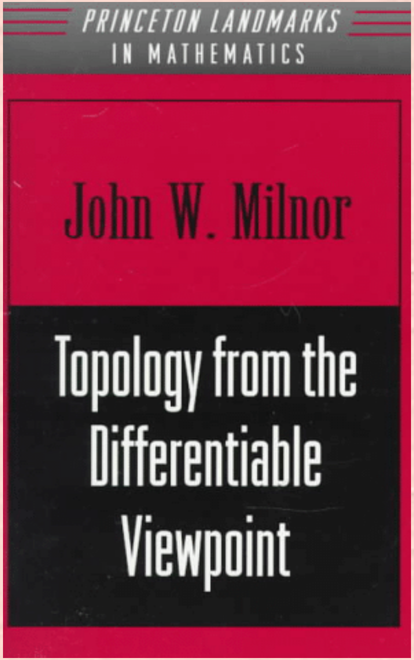
- Continuity and differentiability
- Parametric functions
- Implicit functions

# Continuity

- Let's pause for a moment in our study of derivatives to polish off some unfinished business.

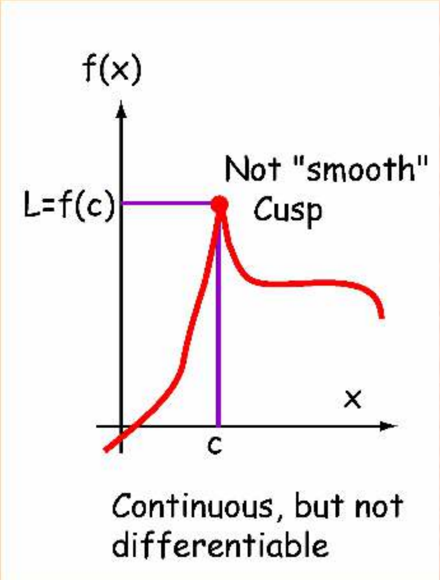
- If a function f has a value for f'(c), then f is said to be differentiable at x = c.

- If f is differentiable at every x value in an interval, then f is said to be differentiable on that interval.



# Continuity

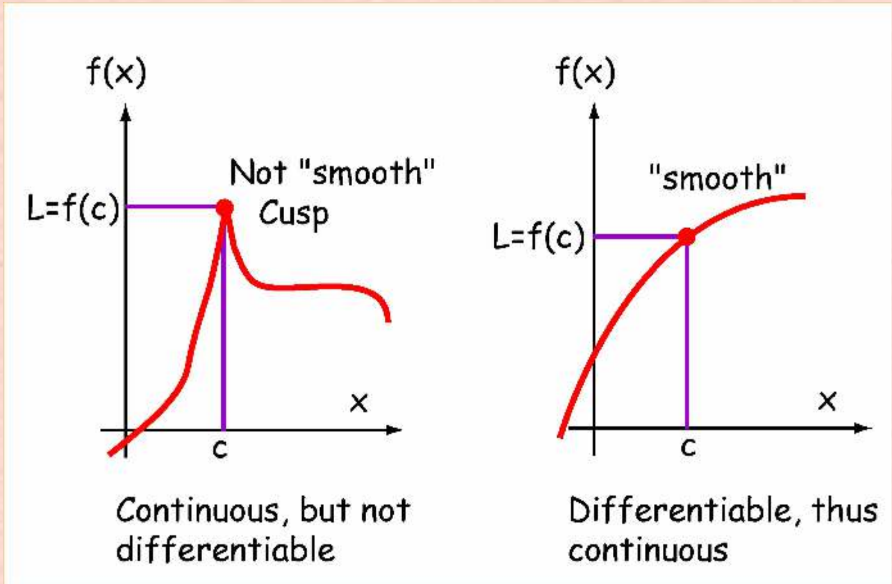
- We saw that a function f is continuous at x =  c if the limit of f(x) as x approaches c equals f(c).




- A function can be continuous at x = c without being differentiable at x = c.

# Continuity

- But a function that is differentiable at $x = c$ is automatically continuous at that point.



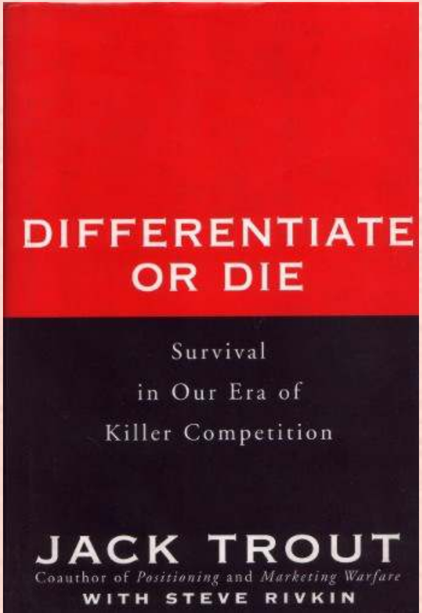

- Let's prove that assertion.

# Continuity

- To show that a function f is continuous at c means showing $\lim_{x \to c} f(x) = f(c)$

- One form of the definition of the derivative already contains all the ingredients:

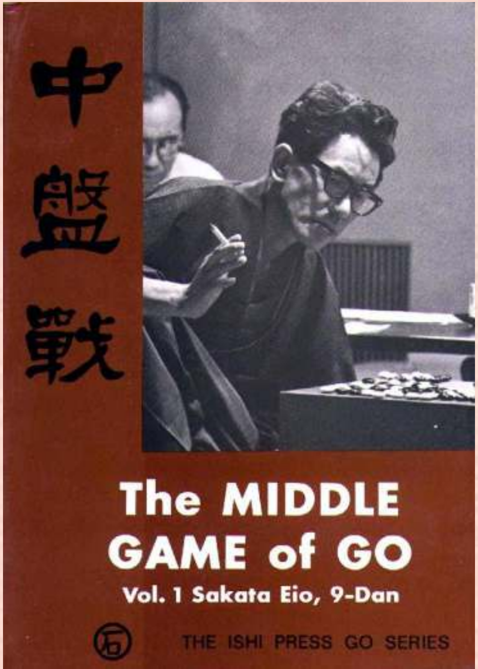
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$



# Continuity

- The game is to perform some operations that lead from the hypothesis (differentiability) to the conclusion (continuity).

- In this case, it's easier (read slicker) to start somewhere "in the middle" and use the hypothesis along the way. Here goes!



# Continuity

$$\lim_{x \to c} [f(x) - f(c)]$$

Start with something that contains limit, f(x) and f(c):

$$= \lim_{x \to c} \left[ \frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$$

Multiply by a slick form of 1

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c)$$

Limit of a product

$$= f'(c) \cdot 0$$

Derivative exists and limit of a linear function

$$= 0$$

# Continuity

So we have

$$\lim_{x \to c} [f(x) - f(c)] = 0$$

$$\lim_{x \to c} f(x) - \lim_{x \to c} f(c) = 0$$

$$\lim_{x \to c} f(x) - f(c) = 0$$

$$\lim_{x \to c} f(x) = f(c)$$ which is the definition of continuity.

- The art of this proof is to multiply by 1 in the form (x-c)/(x-c).
  The rest of the proof involves algebra, limit properties, and the definitions of derivative and limit.

# Continuity

- If function f is differentiable at $x = c$, then f is continuous at $x = c$.

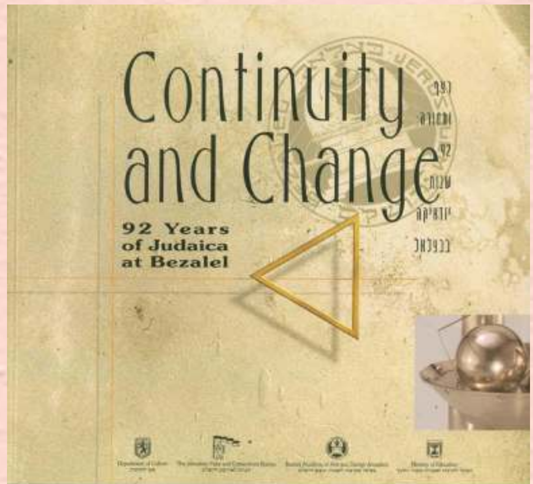
- The contrapositive property is true: if not continuous then not differentiable.

- But the converse (if continuous then differentiable) and
  inverse (if not differentiable then not continuous) properties are false.

# Example

- Prove that $f(x) = x^2 - 7x + 13$ is continuous at $x = 4$.

$f'(x) = 2x - 7$
$f'(4) = 1$, which is a real number.
Therefore f is differentiable at $x = 4$; thus f is continuous at $x = 4$.



# Example

- Is the function $g(x) = (x-2)(x+3)/(x-2)$ differentiable at $x = 2$?

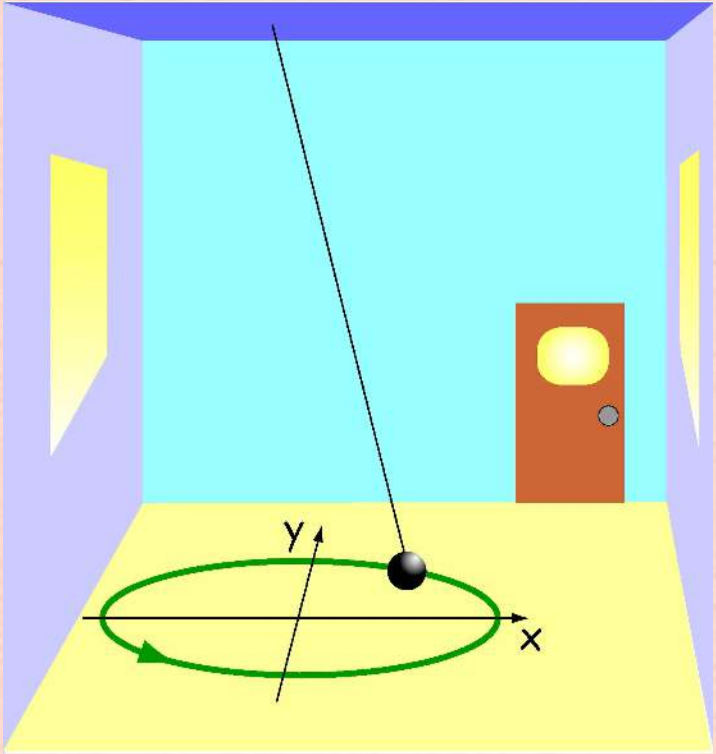
The function has a (removable) discontinuity at $x = 2$, therefore $g$ is not differentiable at $x = 2$.

Lightning in Santa Fe, NM

# Interlude

Continuity
Christine Corda 1998
wax pencil on cardboard

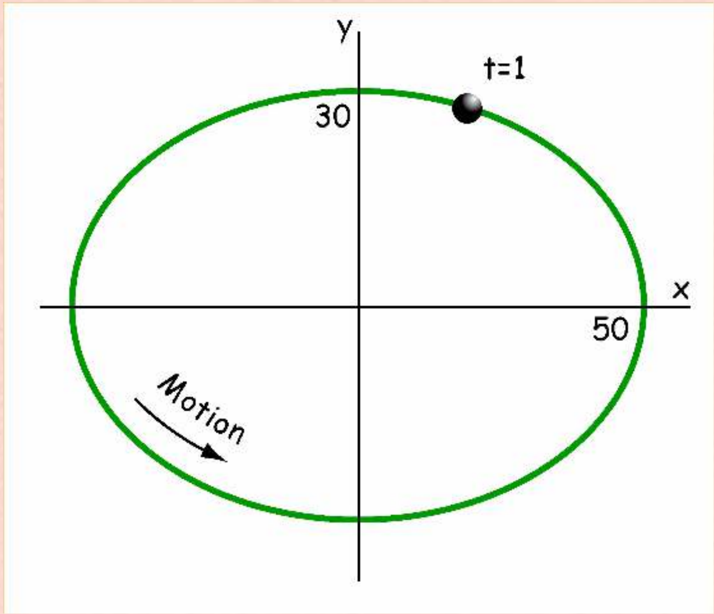
# Parametrics




- Consider a pendulum swinging in both the x- and y-directions.
- Its possible to calculate its velocity not just in the x- and y-directions, but along the curved path.
- We'll use parametric functions to make such determinations.

# Parametrics

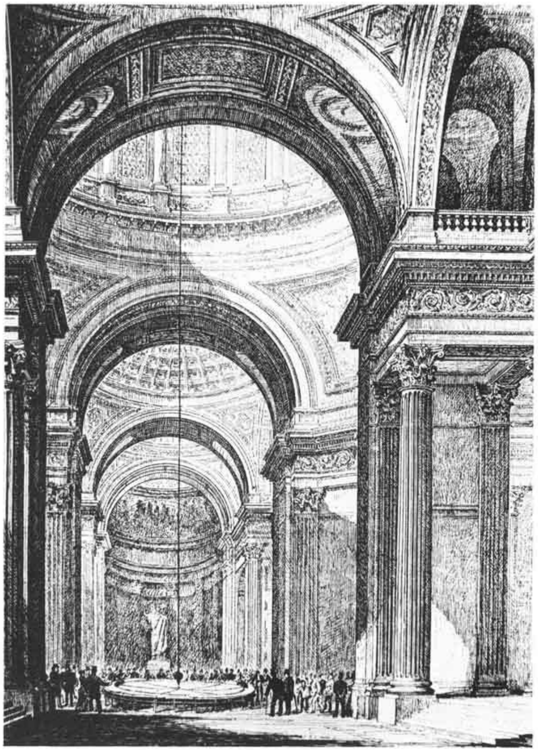
- As the pendulum swings, it goes back and forth sinusoidally in both the x- and y-directions.




- By using the methods we learned a few classes ago, you can find equations for these sinusoids.

# Parametrics

FIGURE 3.17
The Foucault pendulum in the Pantheon, Paris. (Science Museum, London)



- Let the equations of our pendulum be

  $x = 50 \cos 1.2t$

  $y = 30 \sin 1.2t$

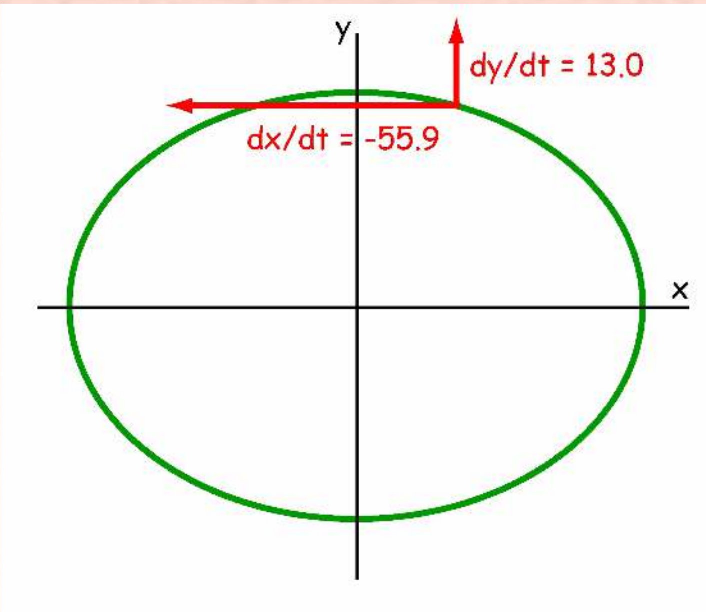
  where x and y are in cm, t in seconds.

- The variable t is called a parameter, parameter meaning “parallel measure”.

- These two equations are called parametric equations.

# Parametrics

- The rates of change in x and y with respect to t can be found by differentiating,



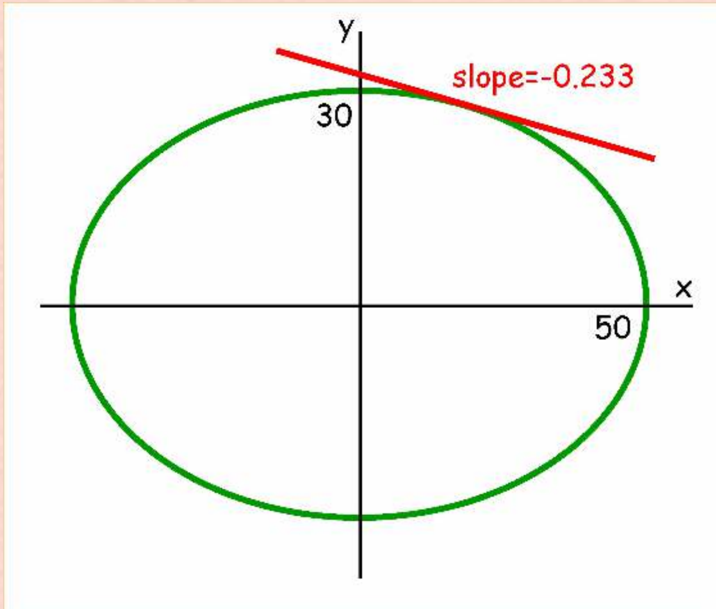

$x = 50 \cos 1.2t$
$dx/dt = -60 \sin 1.2t$

$y = 30 \sin 1.2t$
$dy/dt = 36 \cos 1.2t$

- At t = 1 the pendulum is moving -55.9 cm/s in the x-direction, 13.0 cm/s in the y-direction.

# Parametrics

- If we divide dy/dt by dx/dt, we get the slope of the ellipse dy/dx at t = 1.




$$dy/dx = 13.0/(-55.9)$$
$$= -0.233$$

# Parametrics

- The property illustrated by our pendulum is called the parametric chain rule.

- If x and y are differentiable functions of t, the rate of change of y with x is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- If these were fractions, you could think of the dt terms cancelling each other.

# Example

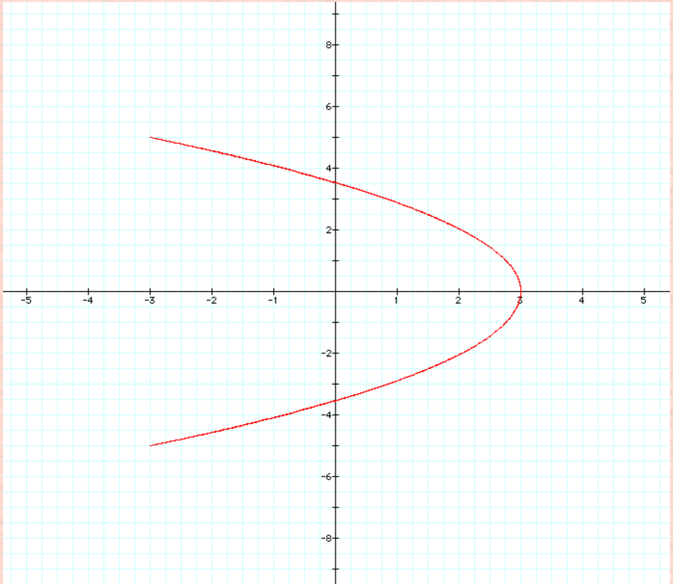
- Given: $x = 3 \cos 2\pi t$
  $y = 5 \sin \pi t$

a) Find a range of t that generates at least one complete cycle of x and y. Plot the graph.

The period for
x is $2\pi/2\pi = 1$.

For y its $2\pi/\pi = 2$.

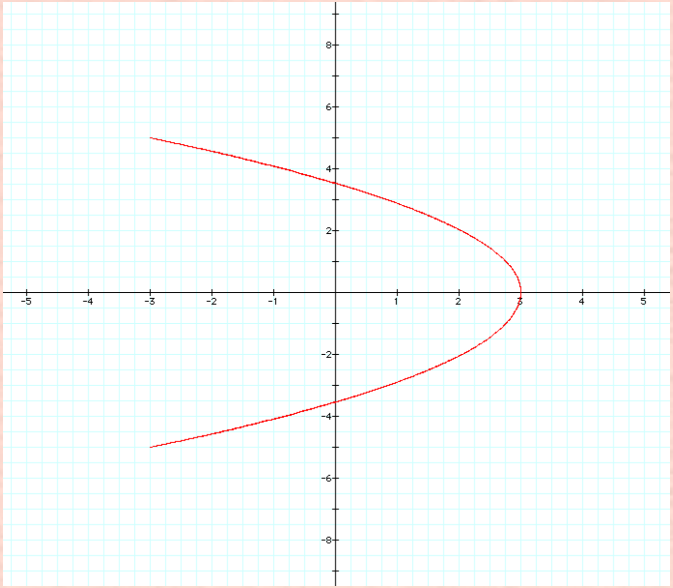
So, $0 < t < 2$ lets
x complete two cycles and
y one cycle.



# Example

b) Describe the behavior of the graph as t increases.

At t = 0 the graph starts at (3,0).




As t increases a tracker goes upward to the left, retraces the path back to (3,0), goes downward to the left and finally retraces the path back to (3,0) at t = 2.

# Example

c) Find an equation for dy/dx in terms of t.

$x = 3 \cos 2\pi t$ $\qquad$ $y = 5 \sin \pi t$

$dx/dt = -6\pi \sin 2\pi t$ $\qquad$ $dy/dt = 5\pi \cos \pi t$

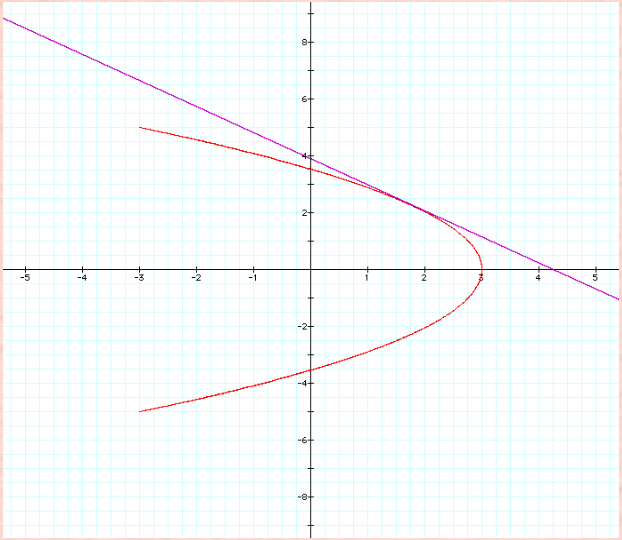
$$dy/dx = (5 \cos \pi t)/(-6 \sin 2\pi t)$$

# Example

d) Find dy/dx when t = 0.15. Show what this means on the graph.

At t = 0.15, $dy/dx = (5 \cos \pi \cdot 0.15)/(-6 \sin 2\pi \cdot 0.15) = -0.917$

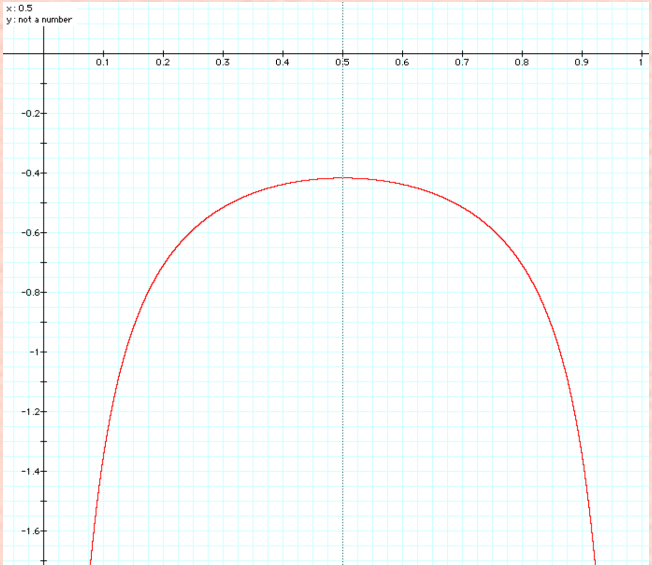
This is the slope of the tangent line at the (x,y) point (1.76, 2.27).



# Example

e) Show dy/dx is indeterminate when t = 0.5.
Find the appropriate limit. How do the answers relate to the graph?

At $t = 0.5$, $dy/dx = (5 \cos \pi \cdot 0.5)/(-6 \sin 2\pi \cdot 0.5) = 0/0$,
which is indeterminate.



# Example

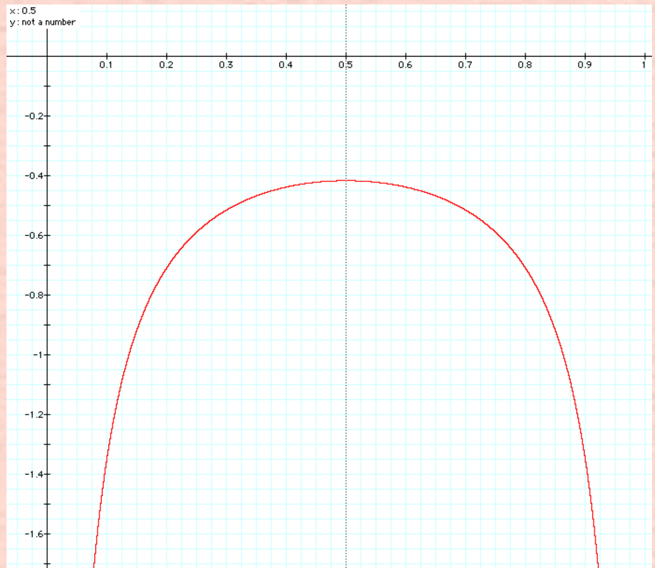
Use l'Hopital's rule (lim f/g = lim f'/g') to find the limit at t = 0.5

dy/dx = lim $(-5\pi \sin \pi t)/(-12\pi \cos 2\pi t)$

= $(-5\pi \sin \pi \cdot 0.5)/(-12\pi \cos 2\pi \cdot 0.5)$

= $(-5\pi)/(-12\pi(-1))$

= $-5/12$

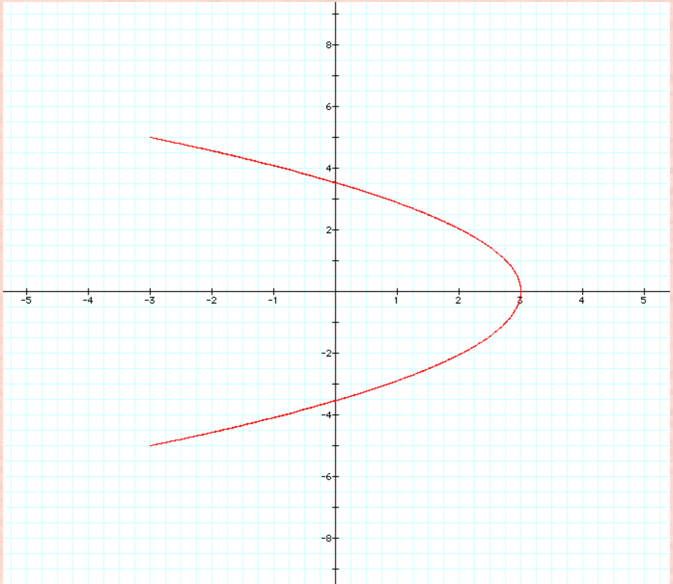


# Example

f) Make a conjecture about the type of geometrical figure.
Then eliminate the parameter t and analyze the resulting equation.

The x-y curve looks like a parabola.

Eliminating the parameter t involves solving one equation for t in terms of x (or y) and substituting the result into the other equation.



# Example

Sometimes there are artful shortcuts, as is the case here.

$$x = 3 \cos 2\pi t \qquad y = 5 \sin \pi t$$

$x = 3(1 - 2 \sin^2 \pi t)$ Use the identity $\cos a = 1 - 2 \sin^2(a/2)$

$x = 3(1 - 2 (y/5)^2)$ Use the $y$ equation for the sine

$x = -6/25\, y^2 + 3$ A parabola opening in the $-x$ direction

# Example

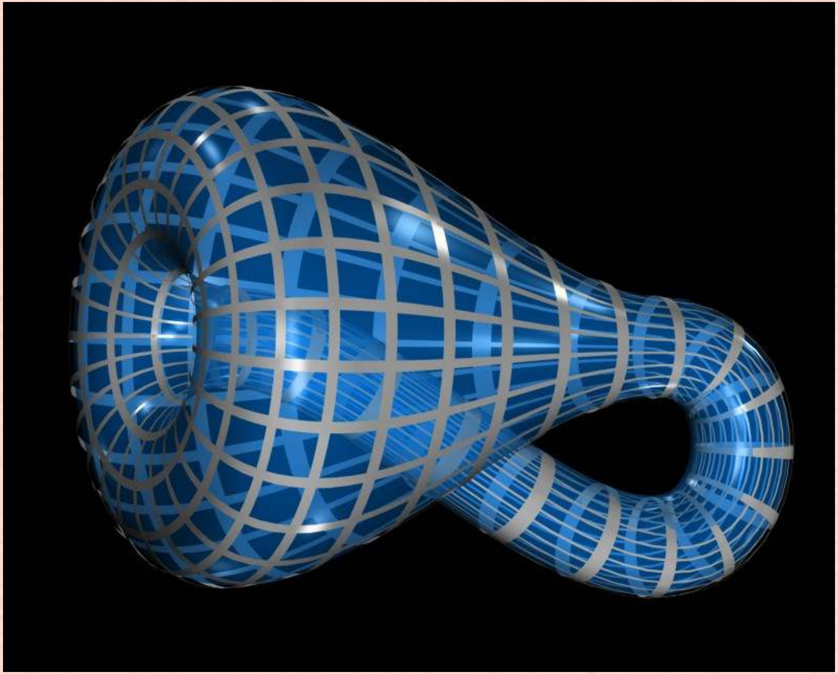
g) Compare the domain of the Cartesian and parametric equations.

$x = -6/25\, y^2 + 3$ $\qquad$ $x = 3 \cos 2\pi t \quad y = 5 \sin \pi t$

The Cartesian equation has an unbounded domain of $x < 3$.

The parametric equations stop at $x = -3$.

# Interlude



$v = 0 \to 2\pi$

$u = 0 \to 2\pi$

$r = 4\,(1 - \cos(u) / 2)$

$$x = \begin{cases} 6\cos(u)\,(1 + \sin(u)) + r\cos(u)\cos(v) & 0 \le u < \pi \\ 6\cos(u)\,(1 + \sin(u)) + r\cos(v + \pi) & \pi < u \le 2\pi \end{cases}$$

$$y = \begin{cases} 16\sin(u) + r\sin(u)\cos(v) & 0 \le u < \pi \\ 16\sin(u) & \pi < u \le 2\pi \end{cases}$$

$z = r\sin(v)$

Most containers have an inside and an outside. A klein bottle is a closed surface with no interior and only one surface.
It is unrealisable in 3 dimensions without intersecting surfaces. It can be realised in 4 dimensions.

# Implicit

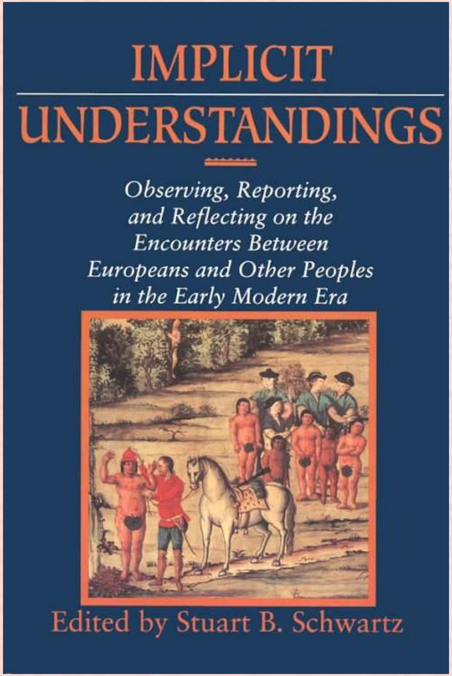
- If y equals some function of x, such as $y = x^2$, then there is said to be an explicit relationship between x and y.

- The word "explicit" comes from the same root as the word "explain".

PARENTAL
ADVISORY
EXPLICIT LYRICS

# Implicit

- If x and y appear in an equation such as $x^2 + y^2 = 25$, then there is an implicit relation between x and y because it is only "implied" that y is a function of x.




- We used implicit differentiation to find the derivatives of the inverse trigonometric functions.

# Implicit

- To find dy/dx for a relation whose equation is implicit:

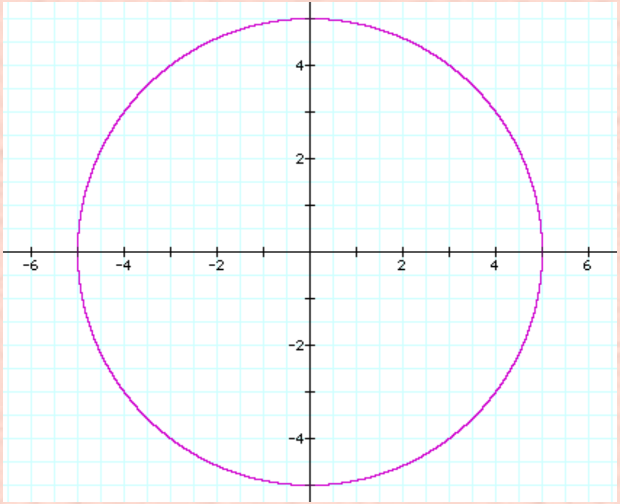
  1. Differentiate both sides of the equation with respect to x. Observe the chain rule by multiplying by dy/dx each time you differentiate an expression containing y.

  2. Use algebra to isolate dy/dx on one side of the equation.

# Example

- Consider $x^2 + y^2 = 25$

a) Tell why the graph is a circle.




The graph is a circle by the Pythagorean theorem, $a^2 + b^2 = c^2$.
All points on the graph are 5 units from the origin, implying the graph is a circle.

# Example

b) Find dy/dx.

$x^2 + y^2 = 25$

$2x + 2yy' = 0$

$2yy' = -2x$

$y' = -x/y$

Note $y'$ is in terms of $x$ and $y$, which is okay because the original relation had $x$ and $y$ together.

# Example

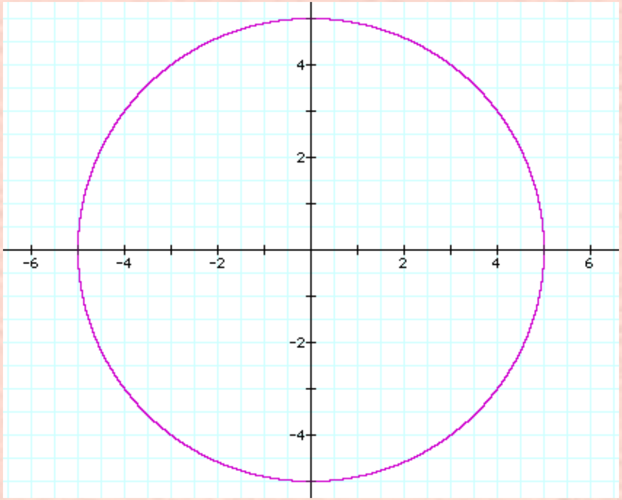
c) Find two values of y when x = 3.

$x^2 + y^2 = 25$

$9 + y^2 = 25$

$y^2 = 16$

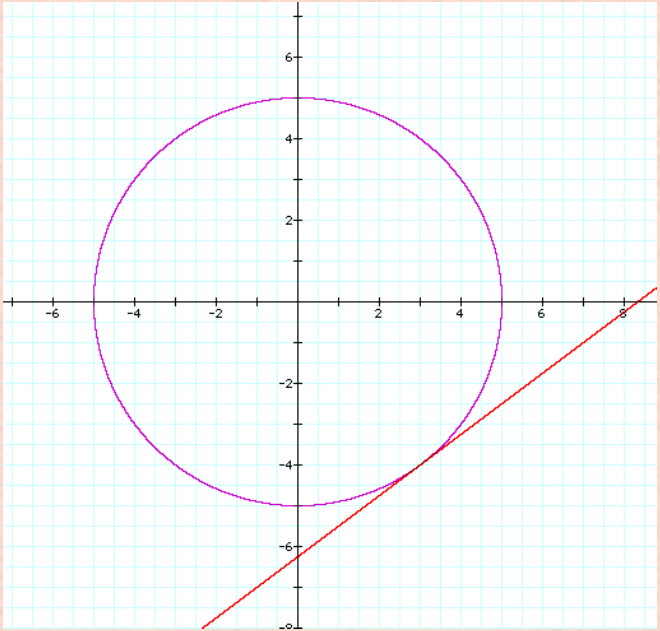
$y = 4$ and $-4$



# Example

d) For the smaller value of y found in part c, find a line with slope dy/dx. How is this line related to the graph?

At the point (3,-4) the derivative is $y' = -x/y = 3/4$.




The equation of the tangent line through this point with this slope is

$$y - y_0 = m(x - x_0)$$

$$y + 4 = 3/4\ (x - 3)$$

$$y = 3/4\ x - 25/4$$

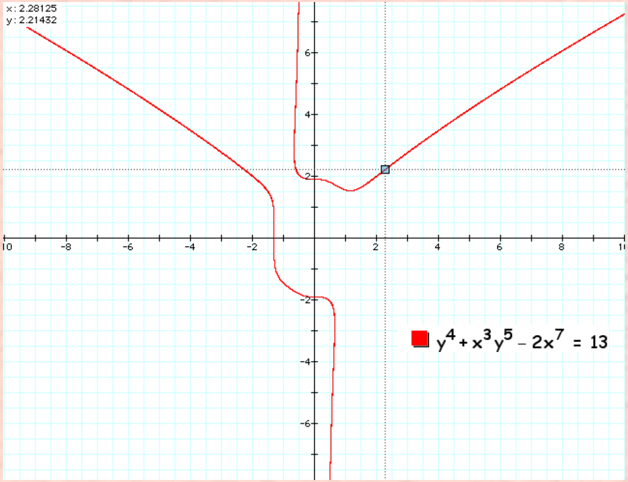
# Example

- For $y^4 + x^3 y^5 - 2x^7 = 13$, find dy/dx.

$$4y^3 y' + 3x^2 y^5 + x^3 \cdot 5y^4 y' - 14x^6 = 0$$

$$y' (4y^3 + 5x^3y^4) = -3x^2 y^5 + 14x^6$$

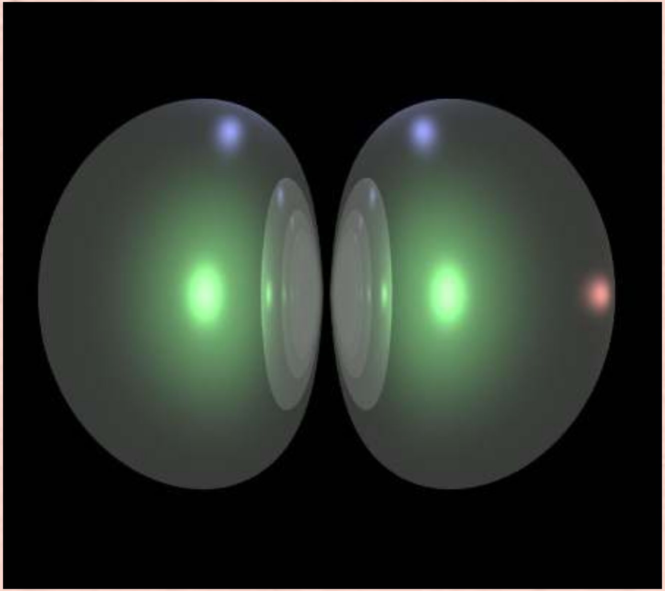
$$y' = ( -3x^2 y^5 + 14x^6 ) \;/\; (4y^3 + 5x^3y^4)$$




- In practice it is usually much harder to find a point on an implicit graph than it is to do the calculus!

# Playtime

- During your in-class problem solving session today you'll examine a few parametric and implicit functions.



P1 atomic orbital
Paul Bourke, 2001

$| x \exp(-0.5 \operatorname{sqrt}(x^2+ y^2+ z^2)) | = 0.1$