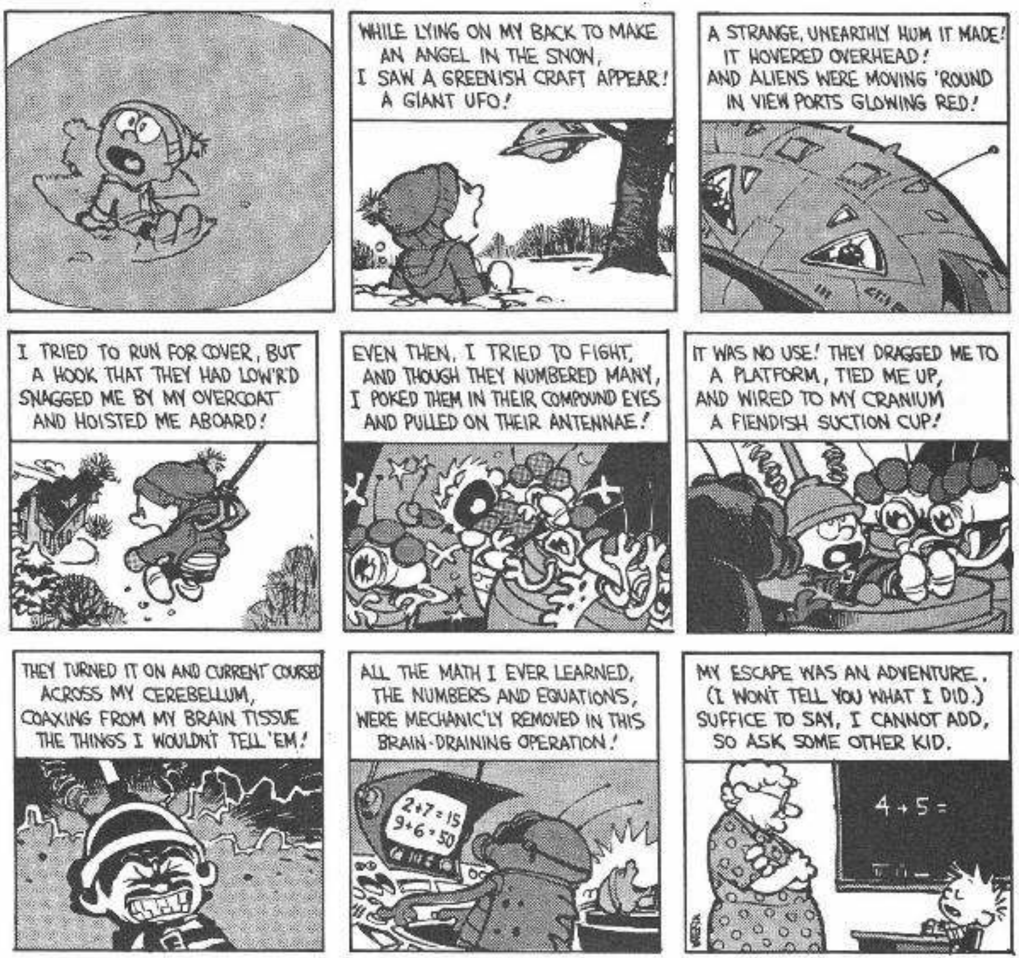
WHILE LYING ON MY BACK TO MAKE AN ANGEL IN THE SNOW, I SAW A GREENISH CRAFT APPEAR! A GIANT UFO!
A STRANGE, UNEARTHLY HUM IT MADE! IT HOVERED OVERHEAD! AND ALIENS WERE MOVING 'ROUND IN VIEW PORTS GLOWING RED!
I TRIED TO RUN FOR COVER, BUT A HOOK THAT THEY HAD LOW'R'D SNAGGED ME BY MY OVERCOAT AND HOISTED ME ABOARD!
EVEN THEN, I TRIED TO FIGHT, AND THOUGH THEY NUMBERED MANY, I POKED THEM IN THEIR COMPOUND EYES AND PULLED ON THEIR ANTENNAE!
IT WAS NO USE! THEY DRAGGED ME TO A PLATFORM, TIED ME UP, AND WIRED TO MY CRANIUM A FIENDISH SUCTION CUP!
THEY TURNED IT ON AND CURRENT COURSED ACROSS MY CEREBELLUM, COAXING FROM MY BRAIN TISSUE THE THINGS I WOULDN'T TELL 'EM!
ALL THE MATH I EVER LEARNED, THE NUMBERS AND EQUATIONS, WERE MECHANIC'LY REMOVED IN THIS BRAIN-DRAINING OPERATION!
2+7=15
9+6=50
MY ESCAPE WAS AN ADVENTURE. (I WON'T TELL YOU WHAT I DID.) SUFFICE TO SAY, I CANNOT ADD, SO ASK SOME OTHER KID.
4+5=

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Calculus

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Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- www.calc101.com
- torte.cs.berkeley.edu:8010/tilu
- archives.math.utk.edu/visual.calculus/3/differentials.2/

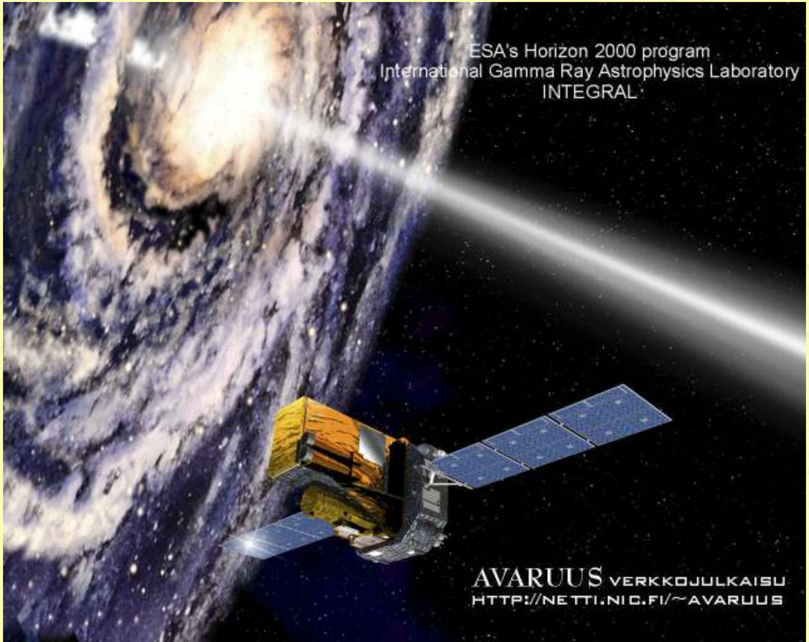
Class #11

- Differentials
- Indefinite Integrals

Integrals

- Recall that a function f is said to be an antiderivative of another function g if and only if $g(x)$ is the derivative of $f(x)$.

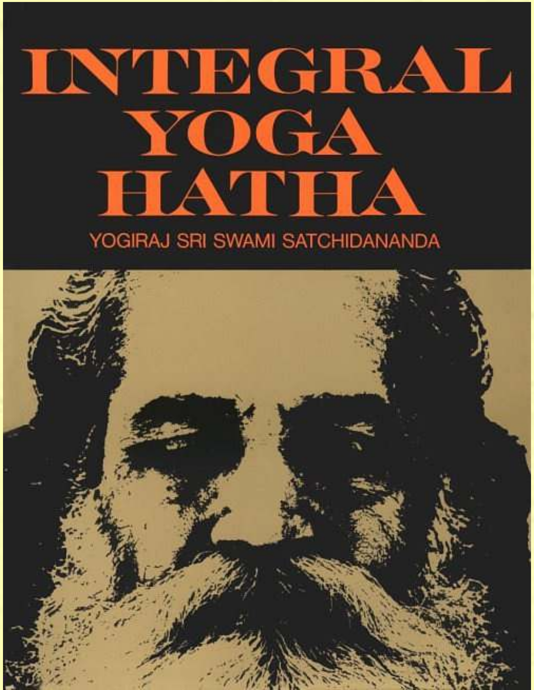
- For instance, if $g(x) = 5x^4$ and $f(x) = x^5$, then f is an antiderivative of g because $f'(x)$ and $g(x)$ are equal.



Integrals

- Note that $x^5 + 7$, $x^5 - 13.2$, and $x^5 + \pi$ are also antiderivatives of $g(x)$ because the derivative of a constant is zero.

- In general, $f(x) + C$ is an antiderivative of $g(x)$, where C is an arbitrary constant called the "constant of integration".



Integrals

- For this reason, an antiderivative is (usually) called an indefinite integral.

- The word "indefinite" is used because there is always a "+ C" whose value isn't determined until an initial condition is specified.

Example

- If $f' = x^7$, find the indefinite integral for $f(x)$.

$f(x) = 1/8\ x^8 + C$ because $f'(x) = x^7$.

- If $f'(x) = x^n$, find the indefinite integral for $f(x)$.

$$f(x) = \frac{1}{n+1} x^{n+1} + C \quad \text{because} \quad f'(x) = x^n$$

Differentials

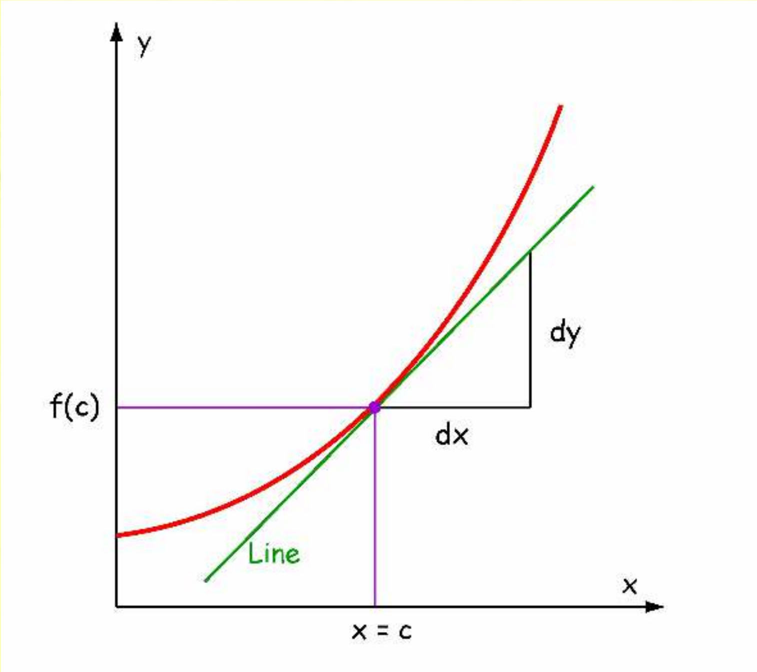
- We've found indefinite integrals by asking "What was differentiated to get this function", but we haven't seen yet a symbol for integration.

- To introduce a symbol for integration, we first need to consider differentials.

Differentials

- Given an equation for a function f and a fixed point on its graph, we want to find an equation for the linear function (the line) that best fits the given function.

- We'll use this linear function to approximate f and values for the differentials dx and dy.

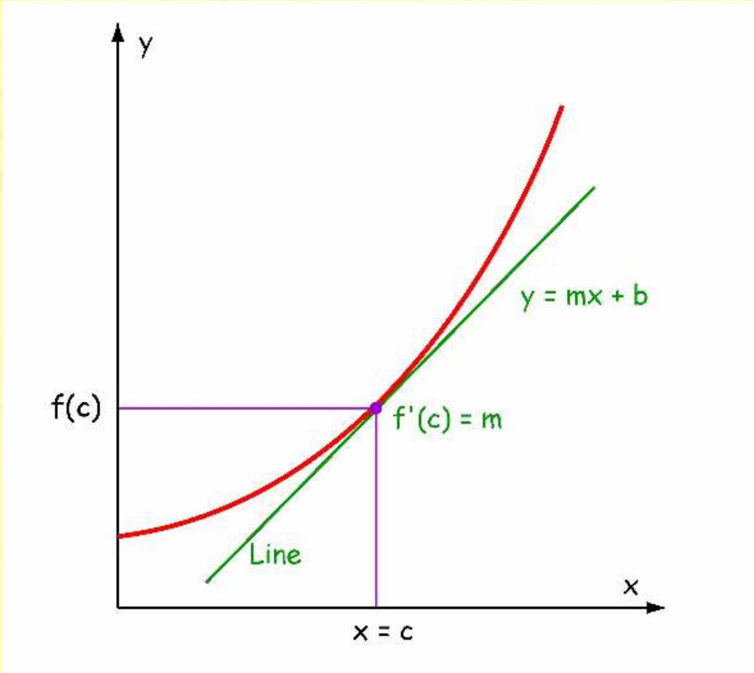


Differentials

- For the line y = mx + b to fit a function close to x = c, two criteria should be met.

- The function values should be equal at x = c, that is, y(c) = c.

- The slopes should be equal at x = c, that is, y′(c) = f′(c).



Example

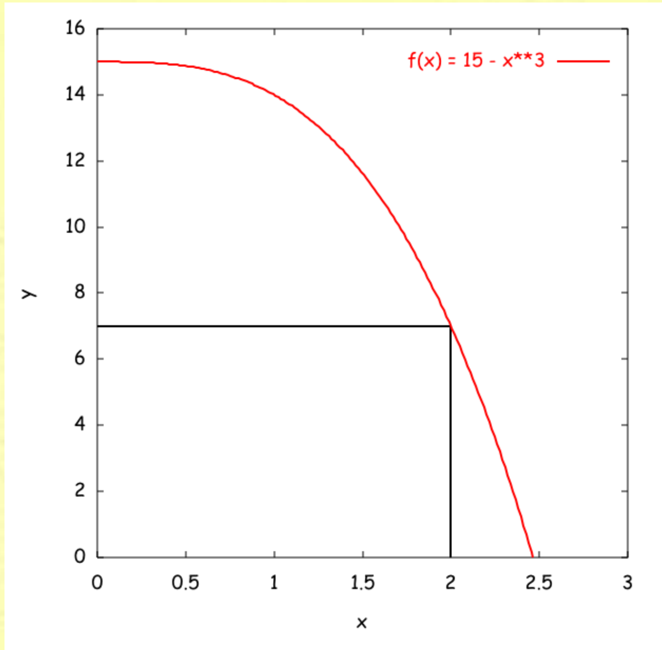
- For $f(x) = 15 - x^3$, find an equation for the line that best fits f at x = 2.
  Find the error in approximating f(x) with the line for values of x close to 2.

$f(2) = 15 - 2^3 = 7$,

so y = 7 when x = 2.

$f'(x) = -3x^2$, $f'(2) = -12$,

so the slope is m = -12 when x = 2



Example

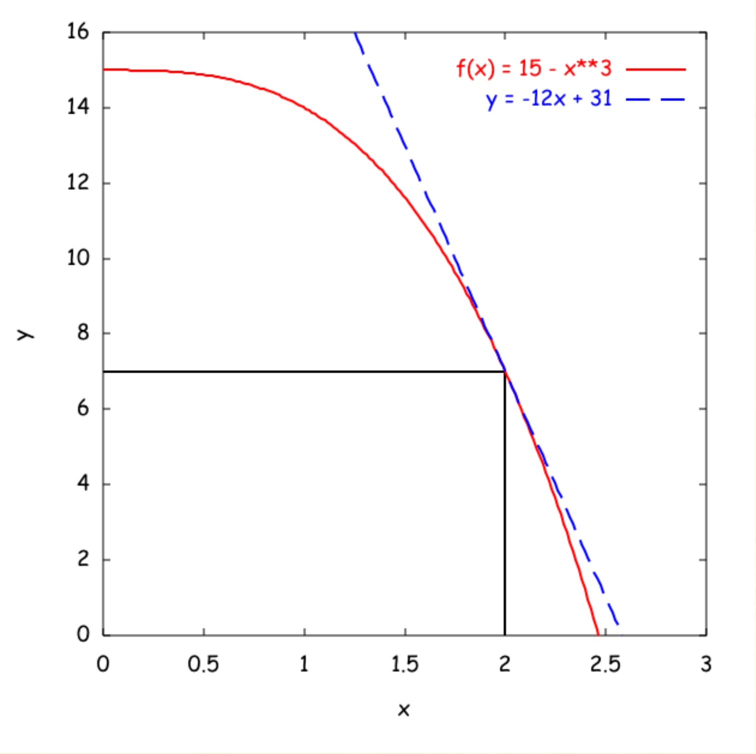
So, the equation of the line is

$$y - y_0 = m\,(x - x_0)$$

$$y - 7 = -12\,(x - 2)$$

$$y = 7 - 12\,(x - 2)$$

$$y = -12x + 31$$



Example

- Tabulating the function f, the line y, and the error f(x) - y gives

x	f(x)	y	error
1.7	10.087	10.6	-0.513
1.8	9.168	9.4	-0.232
1.9	8.141	8.2	-0.059
2.0	7.000	7.0	0.000
2.1	5.739	5.8	-0.061
2.2	4.352	4.6	-0.248
2.3	2.833	3.4	-0.567

- The linear function y fits f(x) perfectly at x = 2, and approximates f, when x is close to 2.

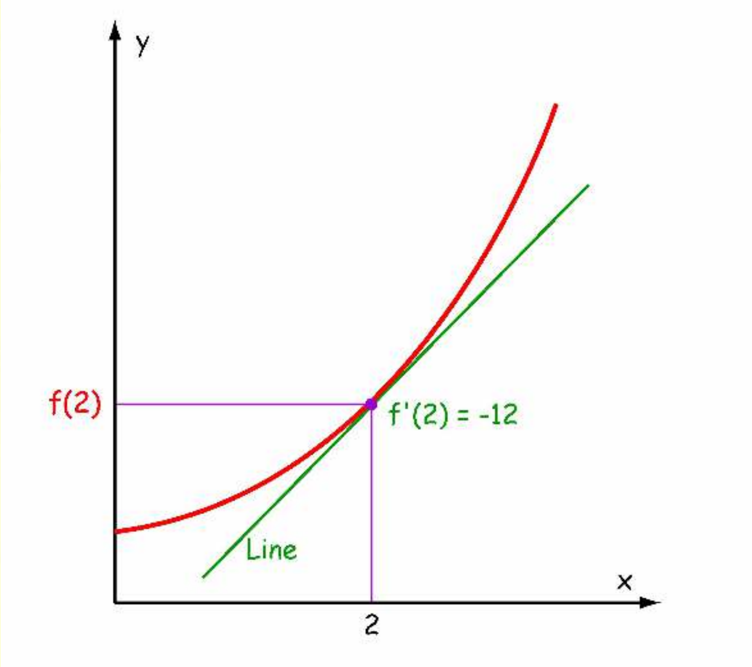
- The error gets larger in absolute value as x gets farther from 2.

Differentials

- The pieces of the linear equation $y = 7 - 12\,(x - 2) = -12x + 31$ are significant:

7 is f(2), the value of the function at the fixed point.

-12 is f′(2), the value of the derivative there.

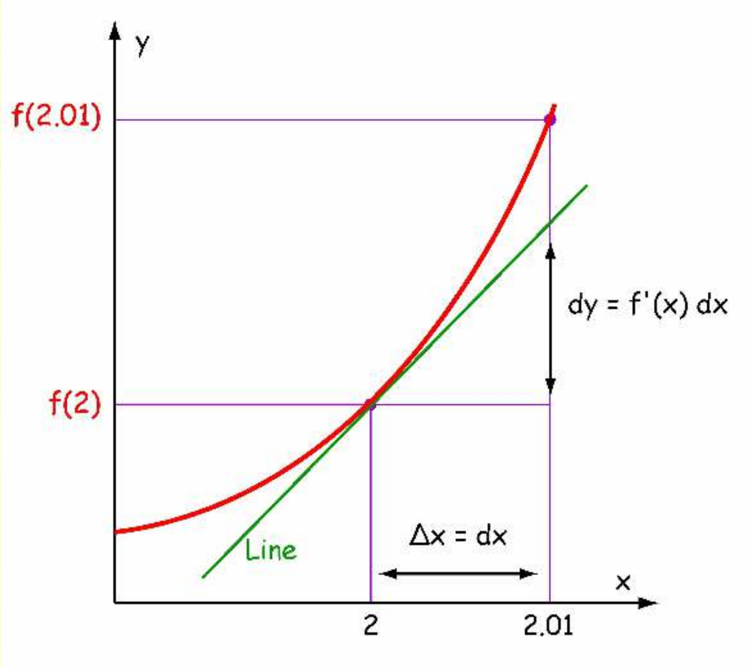


Differentials

- The pieces of the linear equation y = 7 - 12 (x - 2) = -12x + 31 are significant:

(x-2) is the differential dx, the x-distance from the fixed point.

-12(x-2) is the differential dy, the y-displacement for the linear function.



Differentials

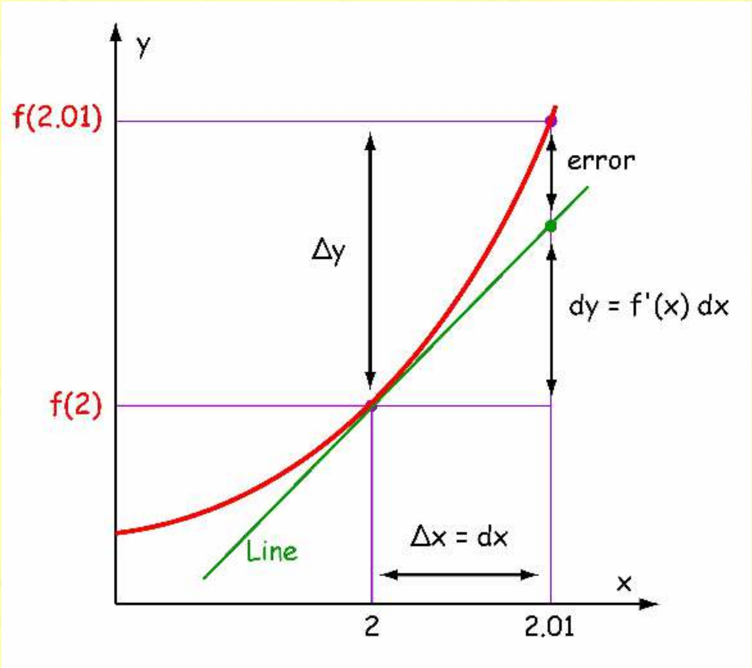
- Our specific example of the differentials dx and dy can be generalized. Thus:

The differentials dx and dy are defined as:

$$dx = \Delta x \quad \text{and} \quad dy = f'(x)\, dx$$

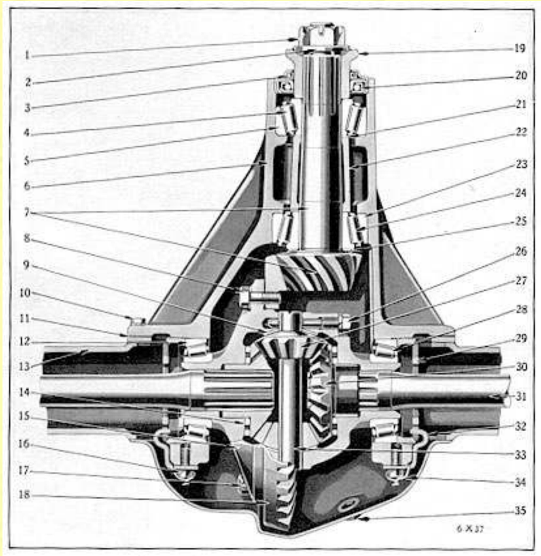
Differentials

- Thus, dy divided by dx is equal to f′(x). Note dy is usually not equal to Δy.



Differentials

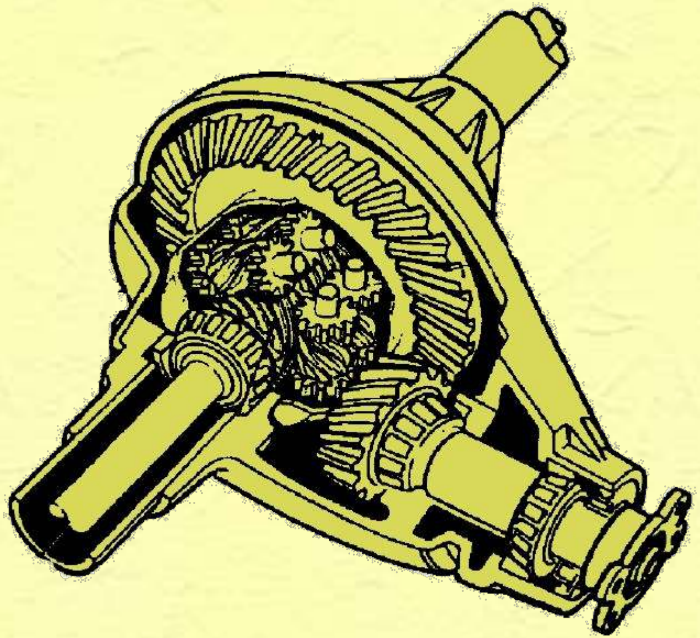
- The most important aspect of the definition of differentials is that it allows for dy and dx in the symbol for the derivative dy/dx to be treated as separate quantities.



Examples

- Find the differential of $w = x^5$.

$$dw = 5\,x^4\,dx$$



Examples

- Find the differential of $z = \tan 4y$.

$$dz = \sec^2(4y) \cdot 4\,dy = 4\sec^2(4x)\,dy$$

Examples

- Find the differential of $p = 3r^2 + \sin r^2$.

$$dp = 3 \cdot 2r\,dr + \cos r^2 \cdot 2r\,dr = (6r + 2r\cos r^2)\,dr$$

Integrals

- Indefinite integration can be considered to be the operation performed on a differential to get the expression for the original function.

- The sign used for this operation is a stretched out S:

$$\int$$

- As we'll see next time, the S shape comes from "sum".

Integrals

- To indicate the we want the indefinite integral of x^5, we write

$$\int x^5 dx$$

- The whole expression, $\int x^5\, dx$, is the integral.
  The function x^5 inside the integral sign is called the integrand.

Integrals

- These words are similar to, for example, the word radical for

$$\sqrt{2}$$

with the radicand being the number 2 inside the square root sign.

$$\int x^5 dx$$

- Although the dx must appear in the integral, "integrating over dx", only the function x^5 is called the integrand.

Integrals

- The expression

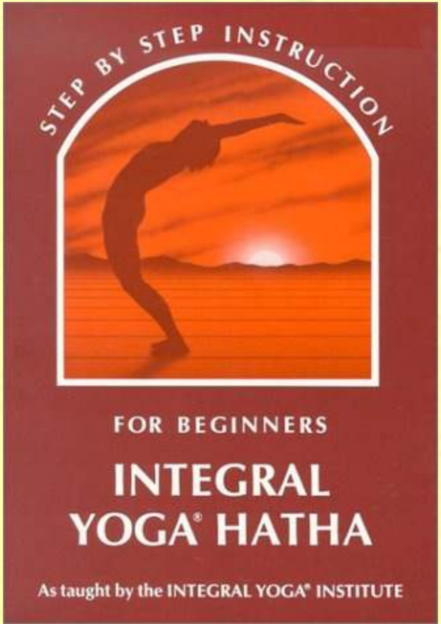
$$\int f(x)\, dx$$

is pronounced "the integral of f(x) with respect to x".
The symbol $\int$ is an operator, like cos or the minus sign, that acts on f(x) dx.

Integrals

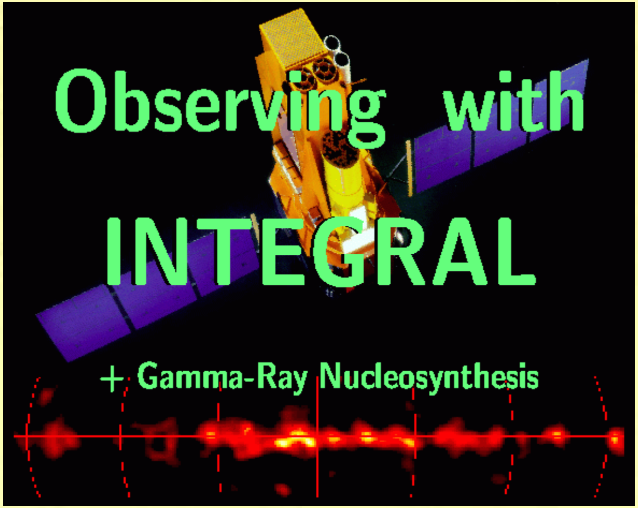
- Hopefully we recognize the integral of a power function:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$



Integrals

- Definition: $$g(x) = \int f(x)\,dx \ \text{ if and only if } \ g(x) = f'(x)$$



Integrals

- A consequence of the definition is integrals and derivatives are inverses:

$$\int \frac{d}{dx} f(x)\, dx = f(x)$$

much like multiplying and then dividing by the same function.

Integrals

- To develop a systematic way of integrating,
  it helps to know some properties of indefinite integrals.

Integrals

- Integral of a constant times a function:

$$\int k\ f(x)\ dx = k \int f(x)\ dx$$

You can pull a constant out through the integral sign.

$$\int 5\cos(x)\ dx = 5\int \cos(x)\ dx = 5\sin(x) + C$$

Integrals

- Integral of a sum of two functions:

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Integral of a sum is the sum of the integrals.

$$\int (x^5 + \sec^2 x - x)\, dx = \int x^5 dx + \int \sec^2 x\, dx - \int x\, dx$$

$$= \frac{1}{6} x^6 + \tan x - \frac{1}{2} x^2 + C$$

Integrals

- It doesn't matter which letter you use for the variable of integration:

$$\int \cos x \, dx = \sin x + C$$

$$\int \cos r \, dr = \sin r + C$$

$$\int \cos z \, dz = \sin z + C$$

- The words "with respect to" identifies the variable in the integrand.
  This is of great utility when integrating composite functions.
  For example …

Example

- Evaluate $\int 5 \cos (5x+3)\, dx$

- The idea is to substitute for something in the integrand so that it becomes something we know the integral of, like a cosine function.

$$\text{Let } z = 5x + 3,$$

$$\text{then } dz = 5\, dx$$

$$\int 5 \cos (5x+3)\, dx = \int \cos (5x+3)\, 5\, dx$$

$$= \int \cos z\, dz$$

$$= \sin z + C$$

$$= \sin(5x+3) + C$$

Voice Systems Integration

Example

- Evaluate $$\int (7x+4)^9 \; dx$$

Again, substitute and try to drive the integral to something easy to integrate:

Let $w = 7x + 4$,

then $dw = 7\ dx$, or $dx = dw/7$.

$$\int (7x+4)^9 \; dx = \int w^9 \; \frac{dw}{7}$$

$$= \frac{1}{7}\int w^9 \; dw$$

$$= \frac{1}{7} \cdot \frac{1}{10} w^{10} + C$$

$$= \frac{1}{70}(7x+4)^{10} + C$$

Playtime

- During your in-class problem solving session today you'll explore some differentials and calculate a few indefinite integrals.

Integral