Mathematical discoveries, small or great, are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared by labour, both conscious and subconscious.

Henri Poincaré

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Calculus

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flash.uchicago.edu/~fxt/class_pages/class_calc.shtml

Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- mss.math.vanderbilt.edu/~pscrooke/MSS/rs.html
- archives.math.utk.edu/visual.calculus/4/riemann_sums.4/
- archives.math.utk.edu/visual.calculus/3/mvt.1/

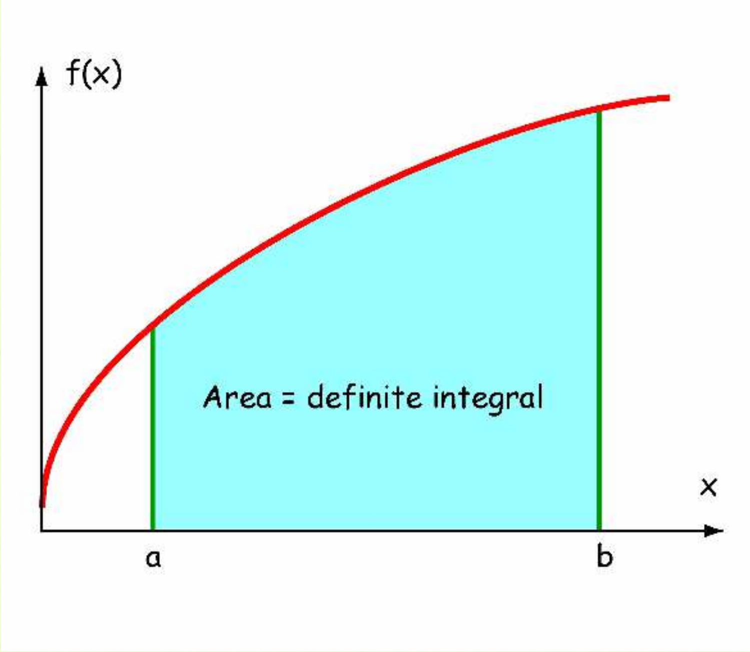
Class #12

- Riemann sums
- Mean value thereom

Sums

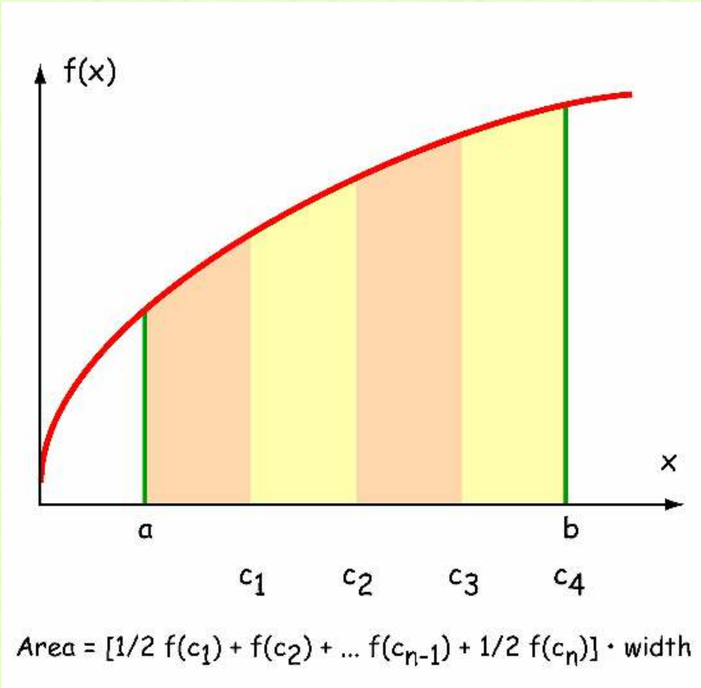
- Recall that a definite integral is used for the product f(x) and x, such as rate • time.

- So, the integral is equal to the area of the region under the graph of f(x).



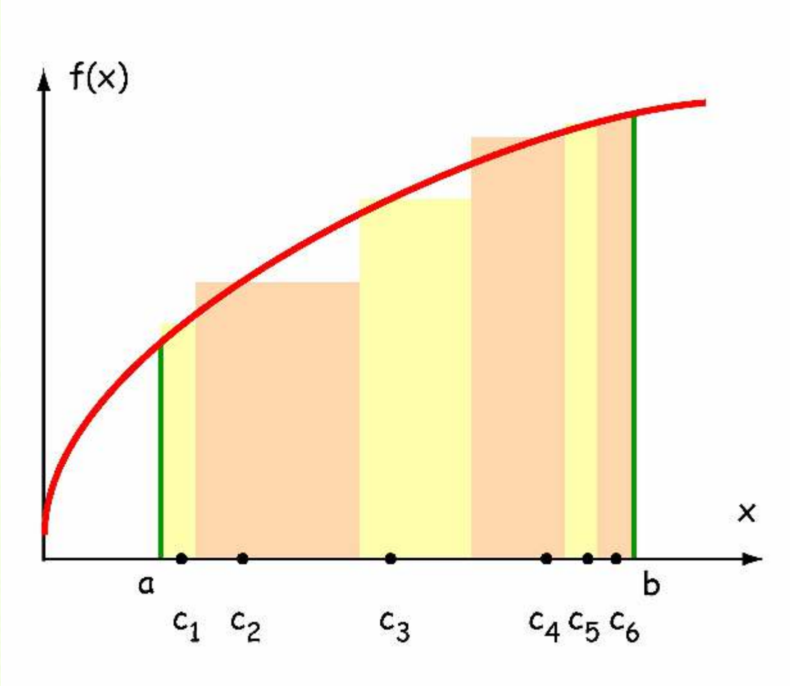
Sums

- Our trapezoid rule let us find definite integrals by slicing the area into strips, then approximating the area of each strip with the area of a trapezoid.



Sums

- Another way to estimate a definite integral is to approximate the strips with rectangles instead of trapezoids.

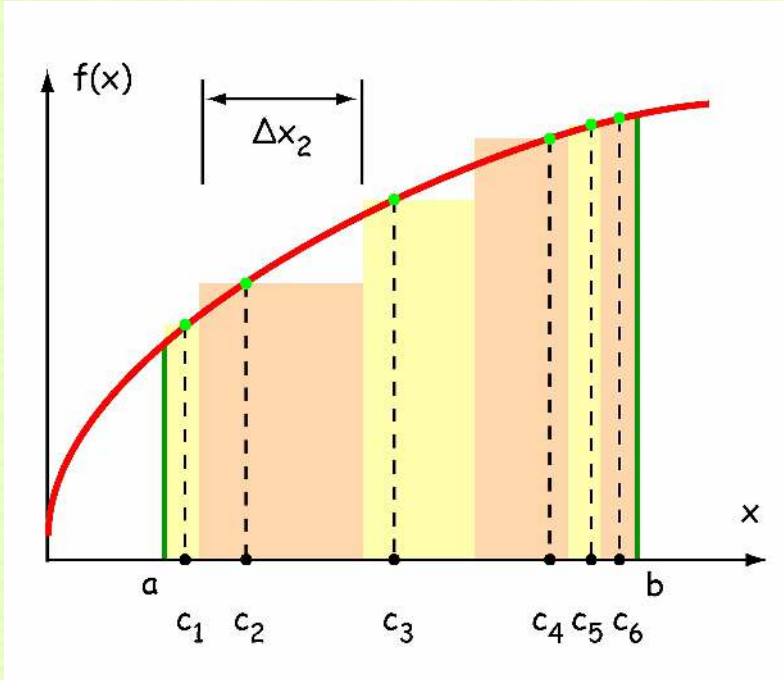


- Values of x (c_1, c_2, c_3 …) are picked so that one value is in each subinterval.
  These x-values are called sample points.

Sums

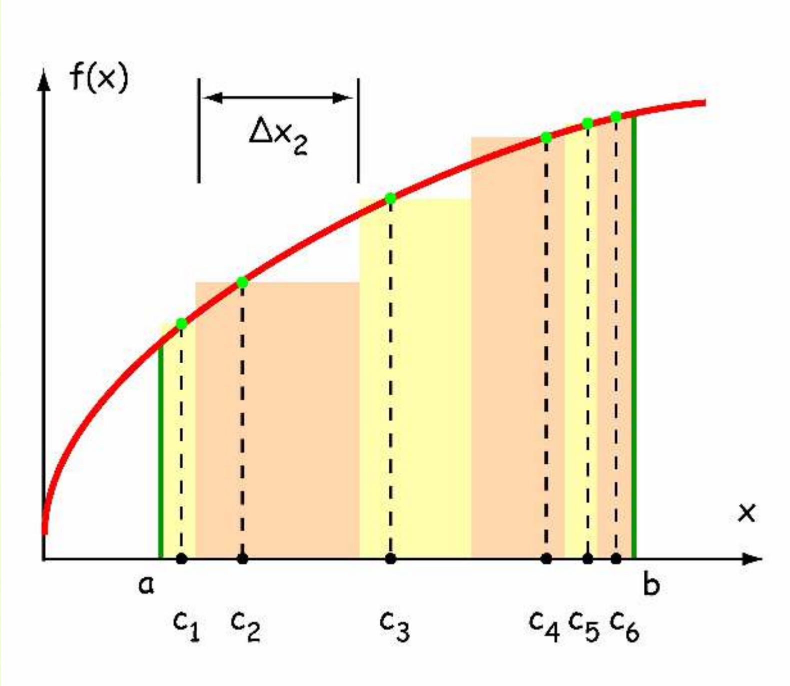
- At each sample point, the corresponding function values $f(c_1)$, $f(c_2)$, $f(c_3)$… are the altitudes of the rectangles.

- The area of any one rectangle is thus
  $A_{rect} = f(c_k)\,\Delta x_k$
  where k = 1,2,3… up to n.



Sums

- The integral is approximately equal to the sum of the areas of the rectangles.



$$\text{Area} \approx \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

Definition

- A Riemann sum, R_n, for a function f on the interval [a,b] is of the form

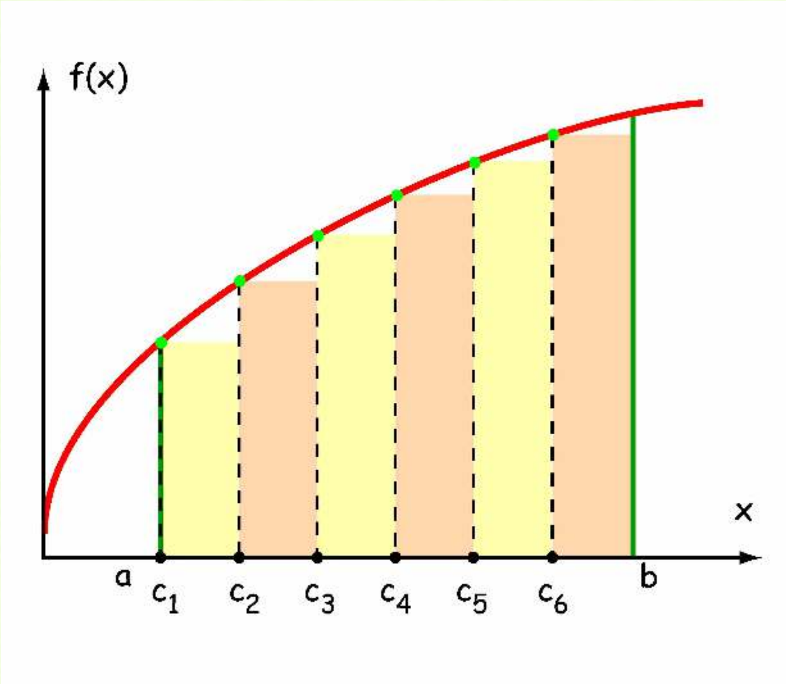
$$R_n = \sum_{k=1}^{n} f(c_k)\, \Delta x_k$$

where the interval [a,b] is partitioned into n subintervals of widths Δk, and the numbers c_k are sample points, one in each subinterval.

Sums

- If each sample point is picked so that each f(c) is the smallest in its subinterval, then each rectangle has an area that is less than the area of the strip.

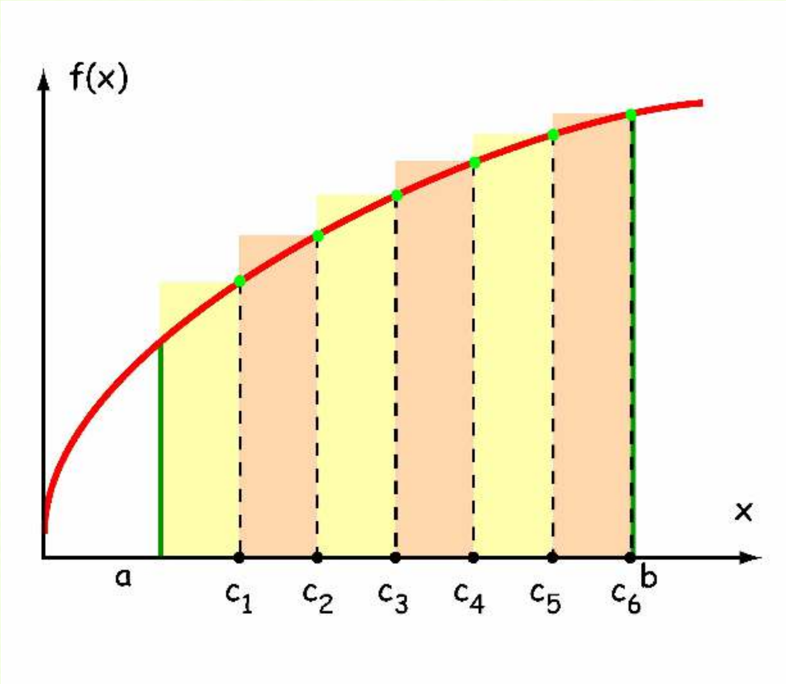
- In this case, the Riemann sum is called a lower sum.



Sums

- Similarly, an upper sum is a Riemann sum with each sample point taken where f(c) is the largest in its respective subinterval.

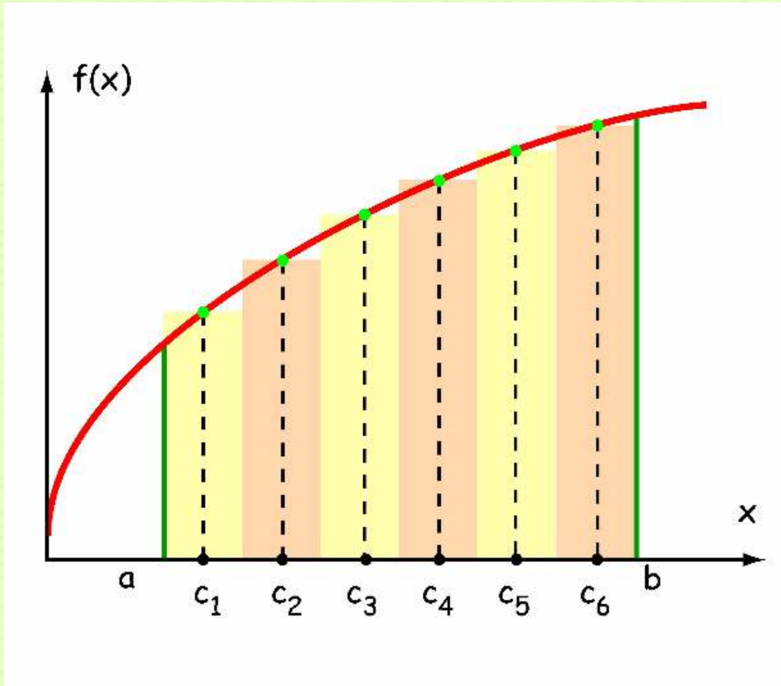
- An upper sum is an upper bound for the area of the region, and a lower sum is a lower bound.



Sums

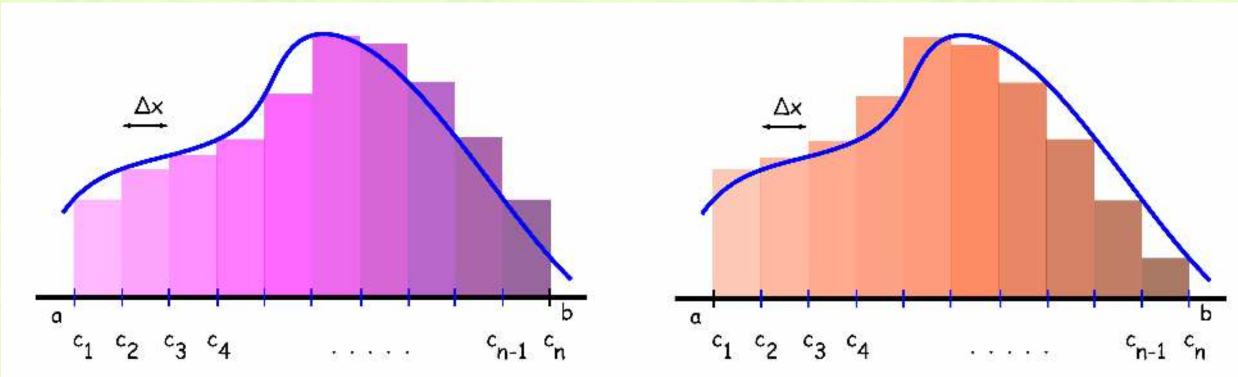
- A midpoint sum is formed by choosing each sample point at the midpoint of the respective area.

- The symbols L_n, U_n, M_n, and R_n are used for the lower, upper, midpoint, and general Riemann sums, respectively.



Sums

- Suppose that the limits of L_n and U_n are equal to each other as the largest value of Δx approaches zero.



- In this case, function f is said to be integrable on the interval [a,b] and the common limit of the upper and lower sums is defined to be the definite integral on the interval [a,b].

Sums

- The integral sign (a stretched S for "sum") with the a and b attached to it is used for a definite integral, like this:

$$\int_a^b$$

- Now, any Riemann sum on [a,b] is bounded by the upper and lower sum.

$$L_n \le R_n \le U_n$$

- So, if function f is integrable on [a,b] then any Riemann sum will also have the definite integral as its limit.

Definitions

- If the lower and upper sums for function f on the interval [a,b] have a common limit as Δx approaches zero, then f is said to be integrable on [a,b].

- This common limit is defined to be the definite integral of f from a to b. The numbers a and b are called the lower and upper limits of integration.

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_k)\,\Delta x_k$$

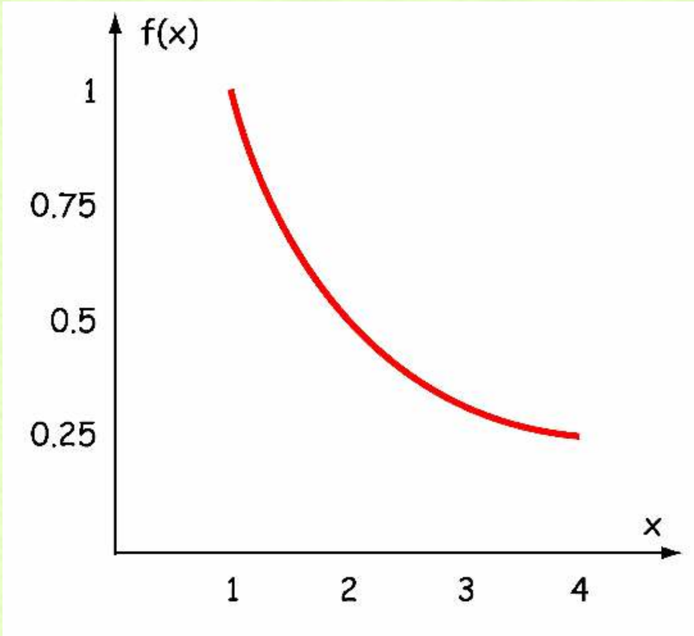
Note the differential dx is used in definite integrals instead of Δx.

Example

- Find L_6, M_6, U_6 for the integral

$$\int_1^4 \frac{1}{x}\,dx$$

For six subintervals, each panel will have a width of $\Delta x = (a - b)/n = (4 - 1)/6 = 0.5$



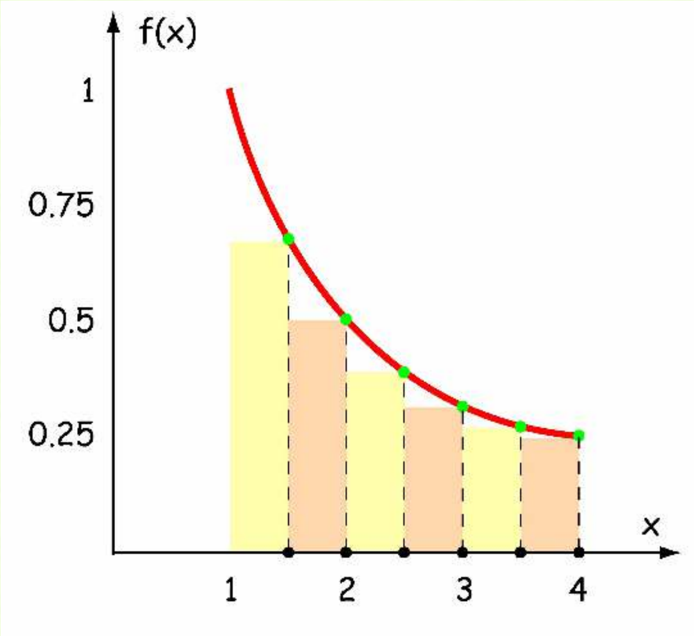
Example

- Since f is decreasing the right ends will give the lowest values, pick them for $L_{6.}$

x	1/x
1.5	0.6666
2.0	0.5
2.5	0.4
3.0	0.3333
3.5	0.2857
4.0	0.25

Sum = 2.4357142

L_6 = Sum • 0.5
= 1.2178571



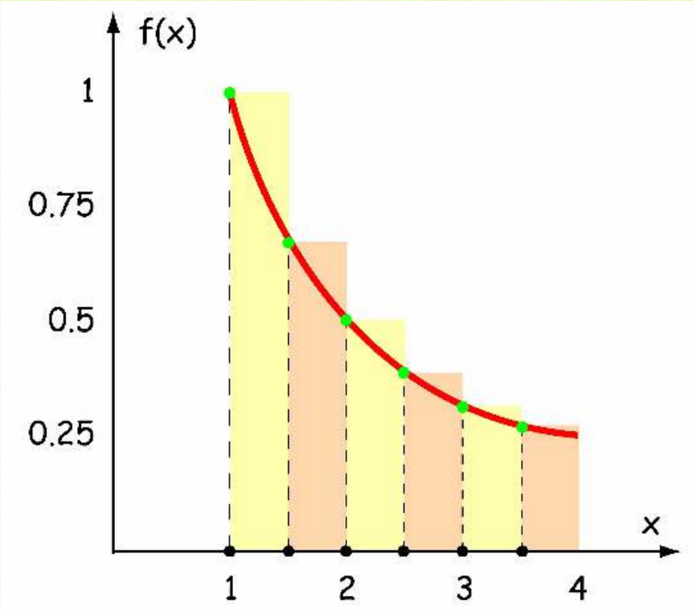
Example

- For the upper sum, pick the sample points at the left ends since f is increasing.

x	1/x
1.0	1
1.5	0.6666
2.0	0.5
2.5	0.4
3.0	0.3333
3.5	0.2857

Sum = 3.1857142

U_6 = Sum • 0.5
= 1.5928571



Example

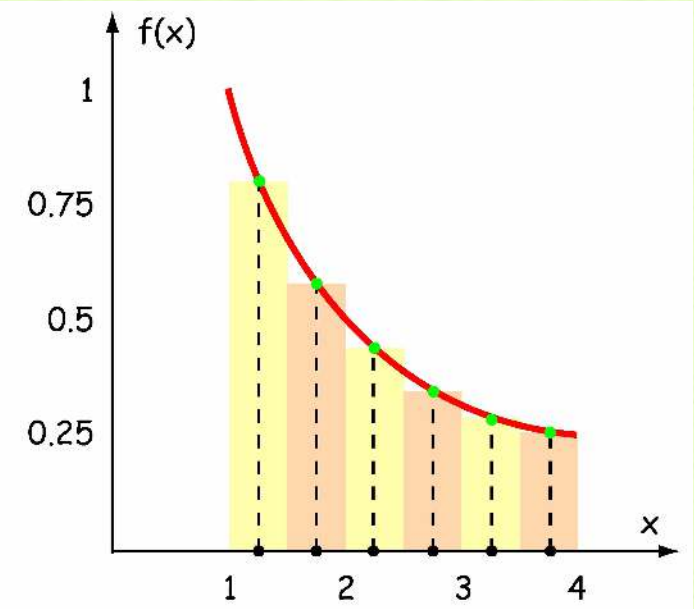
- For the midpoint sum, use sample points 1.25, 1.75, 2.25 … and so on.

x	1/x
1.25	0.8
1.75	0.5714
2.25	0.4444
2.75	0.3636
3.20	0.3076
3.75	0.2666

Sum = 2.7538683…

M_6 = Sum • 0.5

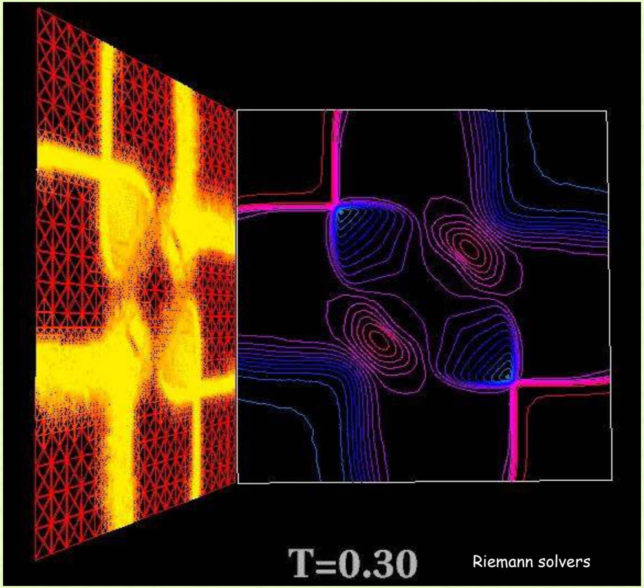
= 1.3769341…



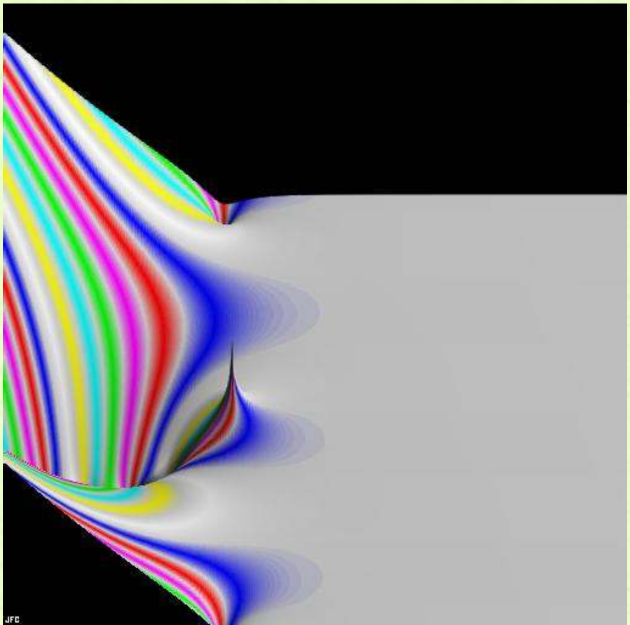
Example

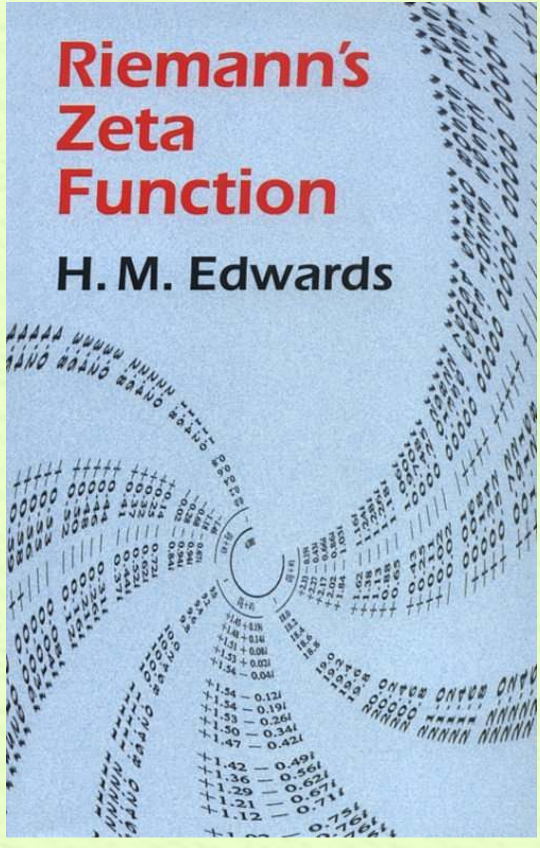
The midpoint sum is between the lower and upper sums,

$$1.2178 < 1.3769 < 1.5928$$



Interlude





Road map

- The approximate value of a definite integral evaluated with Riemann sums depends upon the sample points one chooses - lower, upper, middle, or something else altogether.

- Is there an optimal choice of the sample points?

Mid-points and Peripherals
2000, Howard Buchwald,
acrylic on board,
45 x 40 inches

Road map

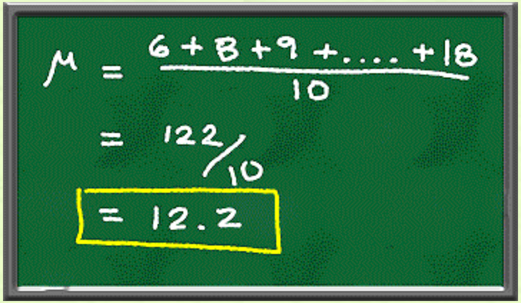
- Yes! The optimal choice of the sample points is embedded within what is called The Fundamental Theorem of Calculus.

- This theorem lets us find definite integrals exactly by using indefinite integrals.

Denali

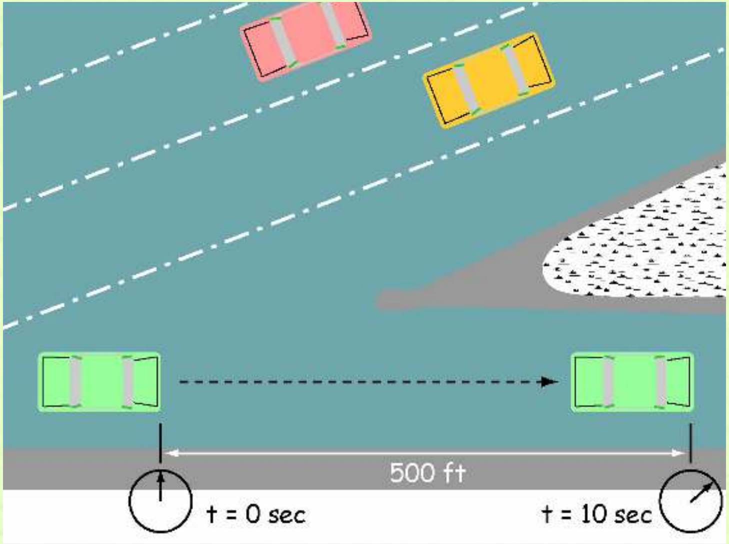
Road map

- So that's were we're headed. But before we get to The Theorem, we need one additional piece of machinery, the mean value theorem.



Mean values

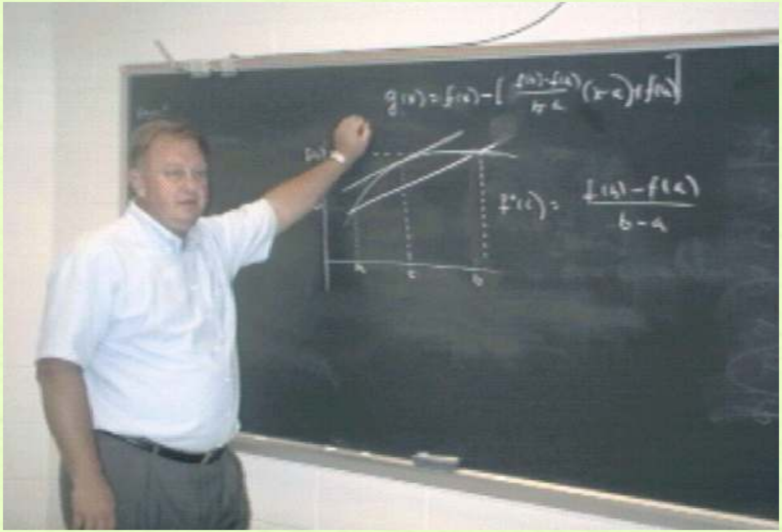
- Suppose that you go 500 ft in 10 sec as you slow down on a freeway exit. Your average speed for the 10 sec interval is 50 ft/s.



- It seems reasonable to conclude that at some point within that 10 sec interval your instantaneous speed was also equal to 50 ft/s.

Mean values

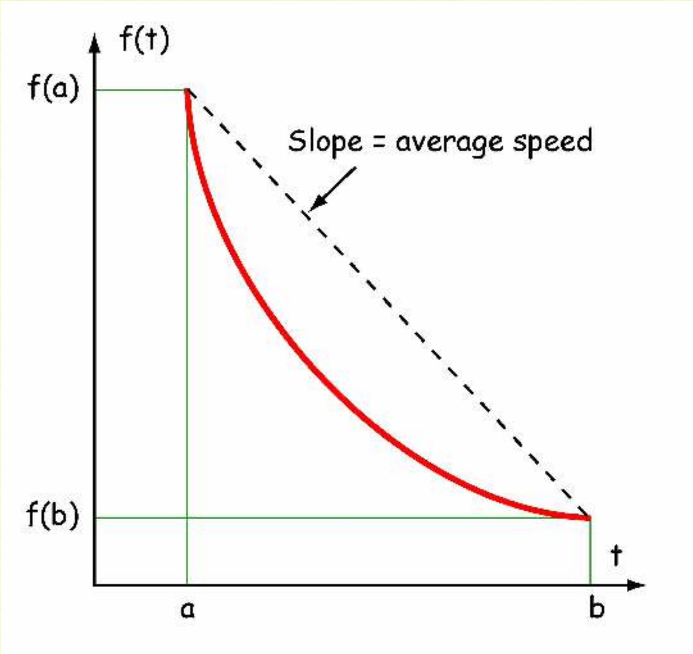
- The mean value theorem states conditions under which this conclusion (an instantaneous value equaling the average value) is true.



Mean values

- Let f(t) be your distance as you exit the freeway. Your average speed between t = a to t = b is

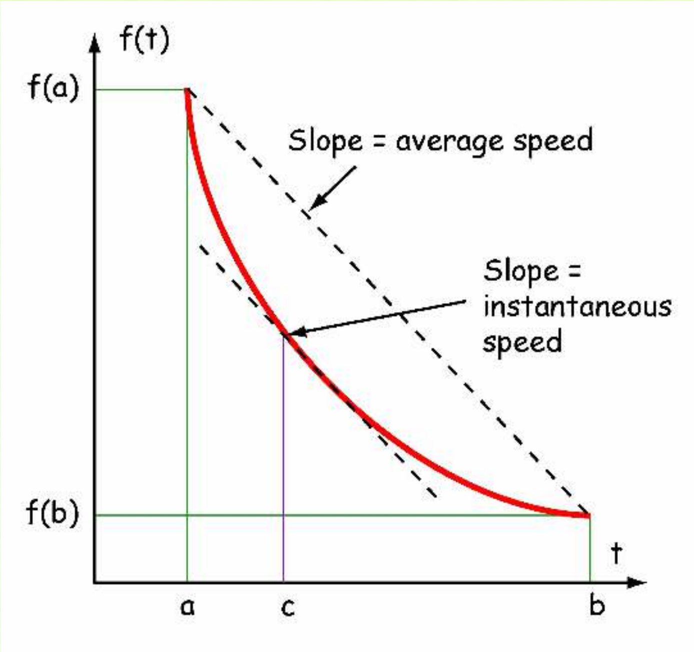
$$\text{Average speed } = \frac{f(b) - f(a)}{b - a}$$



- The average speed is the slope of the secant line connecting a and b.

Mean values

- As we can see from the plot, there is a time t = c between
  a and b at when the tangent line parallels the secant line.



- At this point t = c the instantaneous speed f'(c) equals the average speed.

Mean values

- The mean value theorem gives two sufficient conditions for there to be an instantaneous rate of change equal to the average rate between $x = a$ to $x = b$.

- First, the function must be differentiable for all values of x between a and b.

- Second, the function must be continuous at $x = a$ and $x = b$, even if it's not differentiable at these points.

Mean value theorem

- If (1) f is differentiable for all values of x in the open interval (a,b), and

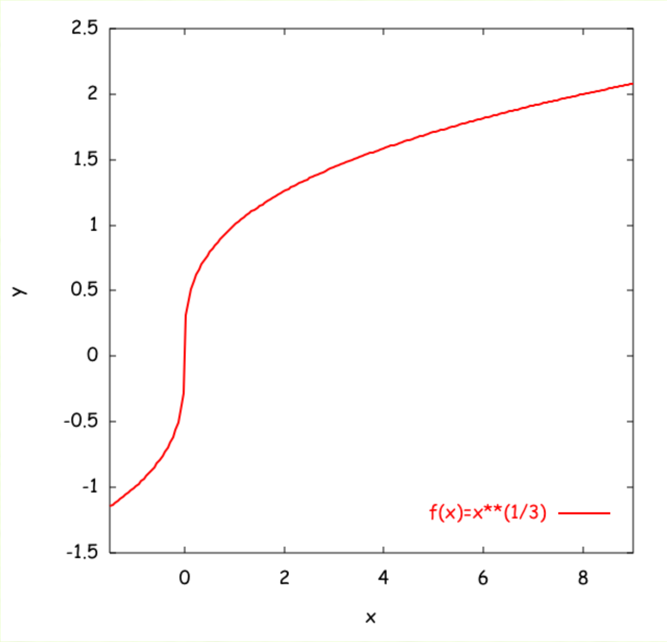
  (2) f is continuous at x= a and x = b,

  then there is at least one number x = c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example

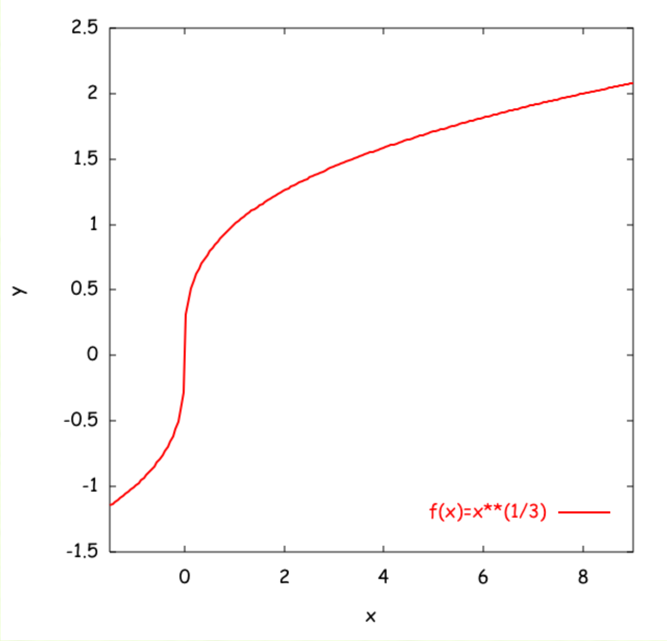
- Given $f(x) = x^{1/3}$, plot the graph. Explain why f satisfies the hypotheses of the mean value theorem on [0,8]



The plot suggests f is differentiable everywhere except at x = 0.

Example

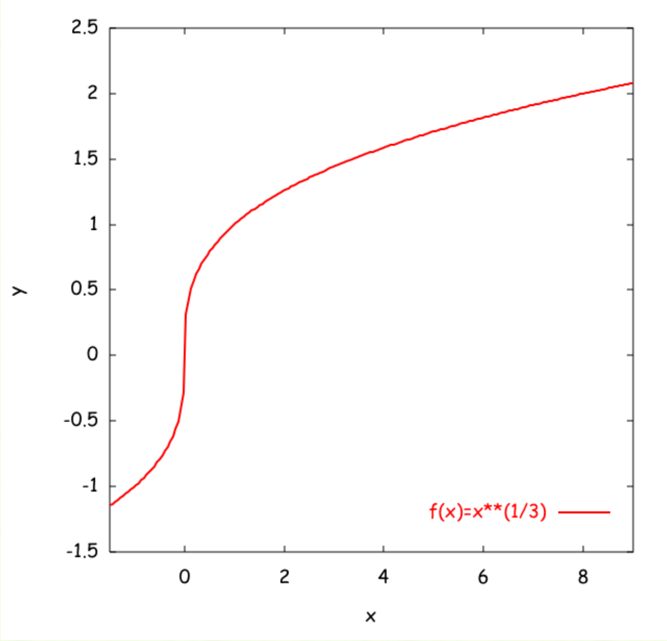
With $f'(x) = 1/3\ x^{-2/3}$,
$f'(0) = 1/0$, which is infinite.



But f is differentiable on the open interval (0,8), a condition of the mean value theorem.

Example

f is continuous at x = 0 and x = 8 because the limits of f(x) as x -> 0 and x -> 8 are 0 and 2, the values of f(0) and f(8), respectively.



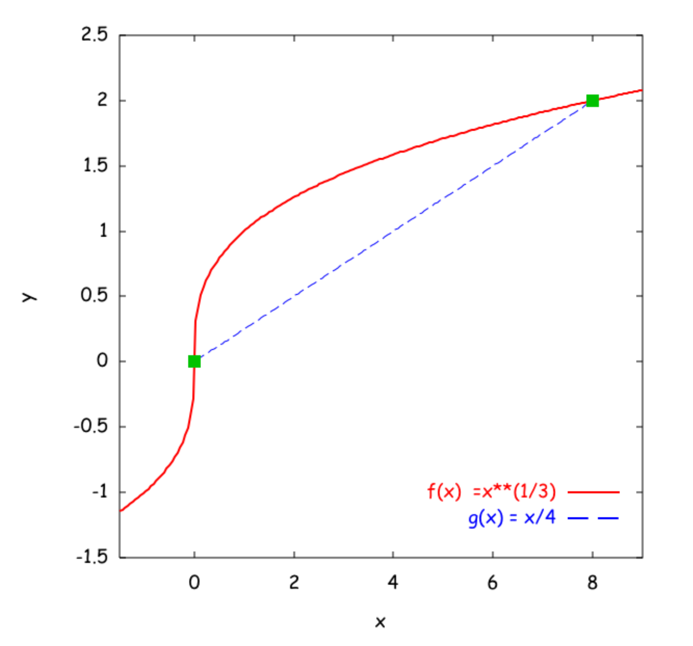
So, the two conditions of the mean value theorem are satisfied.

Example

- Find a value of x in the open interval (0,8) where the mean value theorem is true. Show on the plot that the secant line really is parallel to the tangent line.

The slope of the secant line, the average rate of change, between x = 0 and x = 8 is

$$m_{sec} = \frac{8^{1/3} - 0^{1/3}}{8 - 0} = \frac{1}{4}$$



Example

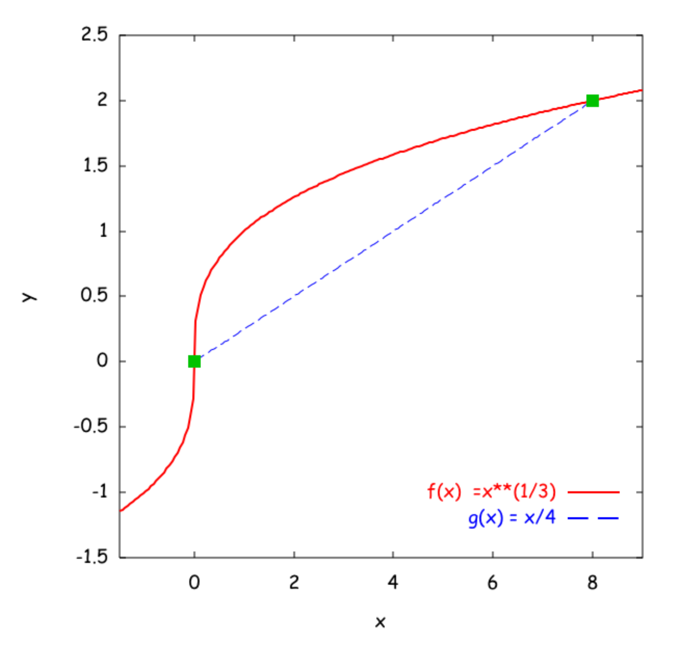
Setting this average slope equal to the instantaneous slope $f'(x) = 1/3\ x^{-2/3}$ at the point x = c, gives

$$\frac{1}{3}c^{-2/3} = \frac{1}{4}$$

$$c^{-2/3} = \frac{3}{4}$$

$$c = \pm\left(\frac{3}{4}\right)^{-3/2} = \pm 1.5396$$

$$c = 1.5396$$



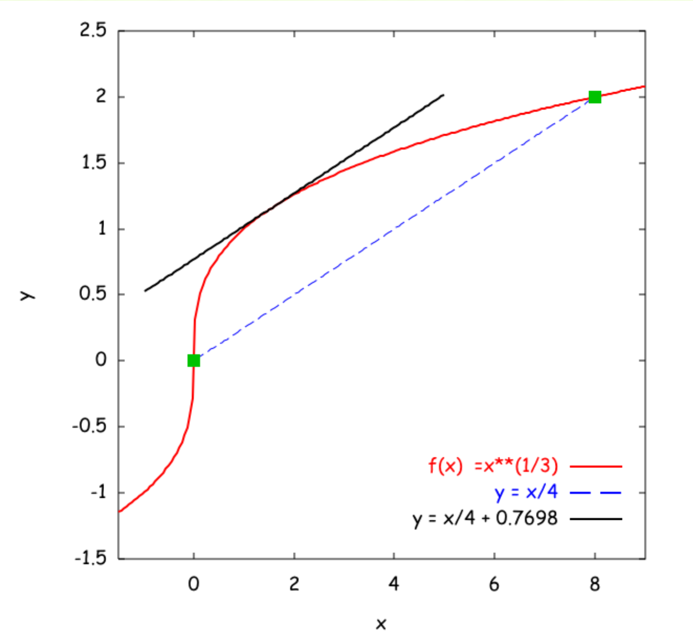
Example

The tangent line that has a slope 1/4 and goes through the point $(c, f(c)) = (c, c^{1/3}) = (1.5396, 1.1547)$ is

$$y - y_0 = m(x - x_0)$$

$$y - 1.1547 = \frac{1}{4}(x - 1.5396)$$

$$y = 0.25x + 0.7698$$



Playtime

- During your in-class problem solving session today you'll do a few Riemann sums and explore the mean value theorem.