The real voyage of discovery consists not in seeking new landscapes, but in having new eyes.

Marcel Proust

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Calculus

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Syllabus

1	Aug 29	Pre-calculus
2	Sept 05	Rates and areas
3	Sept 12	Trapezoids and limits
4	Sept 19	Limits and continuity
5	Sept 26	Between zero and infinity
6	Oct 03	Derivatives of polynomials
7	Oct 10	Chain rule
8	Oct 17	Product rule and integrals
9	Oct 24	Quotent rule and inverses
10	Oct 31	Parametrics and implicits
11	Nov 7	Indefinite integrals
12	Nov 14	Riemann sums
13	Dec 05	Fundamental Theorem of Calculus

Sites of the Week

- archives.math.utk.edu/visual.calculus/4/ftc.9/
- www.ies.co.jp/math/java/calc/perime/perime.html
- mss.math.vanderbilt.edu/~pscrooke/MSS/definiteintegral.html

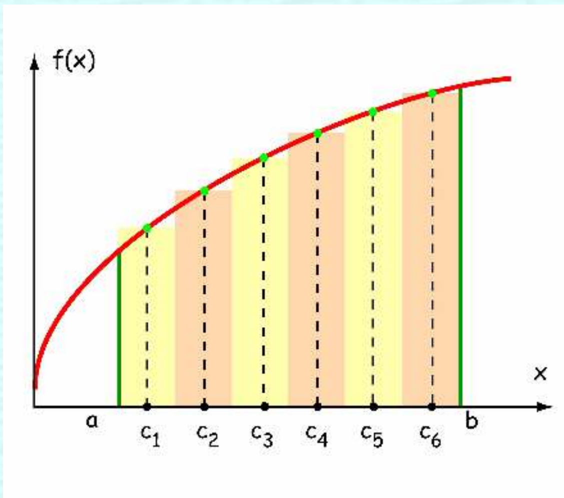
Class #13

- Fundamental Theorem of Calculus

Magic

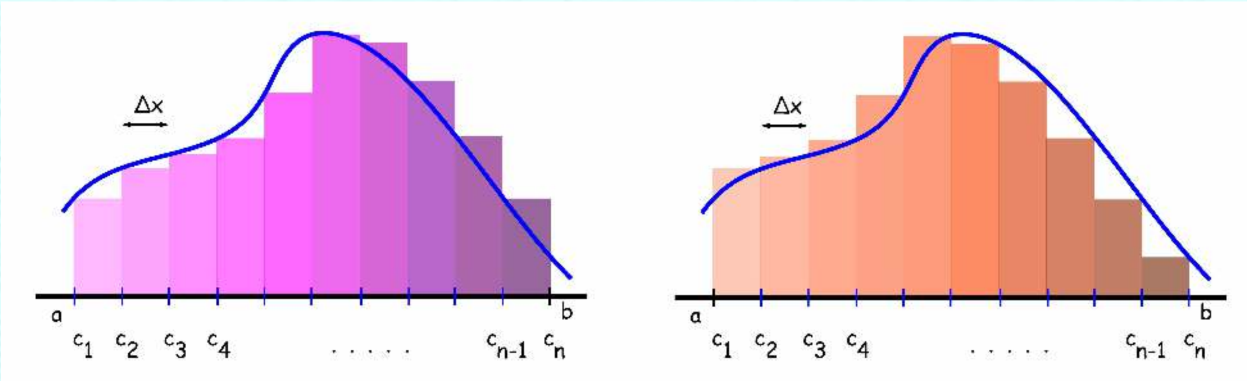
- Riemann sums tend to get closer to the actual value of an integral as the number of intervals increases.

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_k)\,\Delta x_k$$



Magic

- The value of a Riemann sum also depends on just where you pick the sample points c_k in each interval.



- For instance, a midpoint sum should be closer to the actual value of the integral than a sum for which the sample points are taken at one end of the interval.

Magic

- But what if, by magic almost, we could pick the sample points c_k in such a way that the Riemann sum would be independent of the number of intervals?

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_k)\,\Delta x_k$$

- The Riemann sum would then be the same no matter how many or how few intervals we used!

Magic

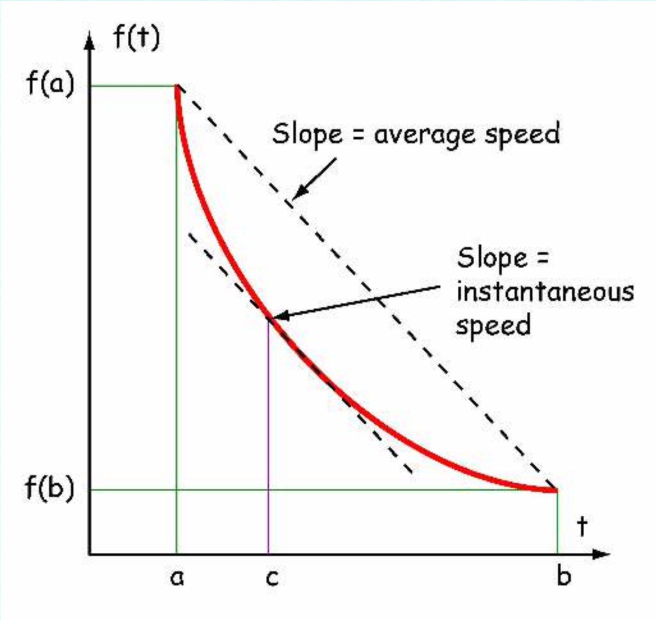
- Such magic is obtained by using the sample points c_k generated from the mean value theorem!

- The end result is called the fundamental theorem of calculus, and it allows us to evaluate indefinite integrals exactly, by using indefinite integrals.

Mean value theorem

- If f(x) is differentiable for all values of x in the open interval (a,b) and f(x) is continuous at x = a and x = b, then there is at least one number x = c in (a,b) such that

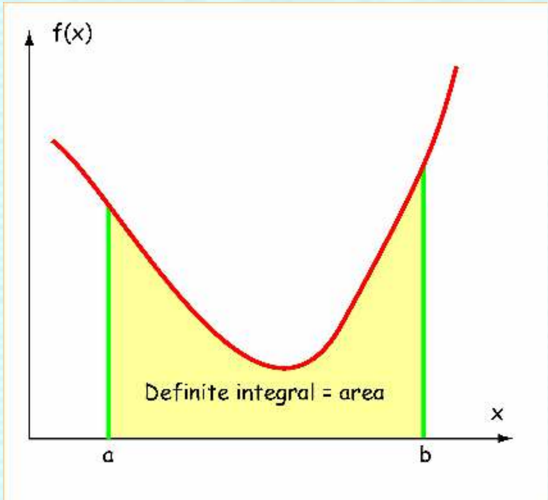
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



To the summit

- We wish to find the definite integral

$$\int_a^b f(x)\,dx$$



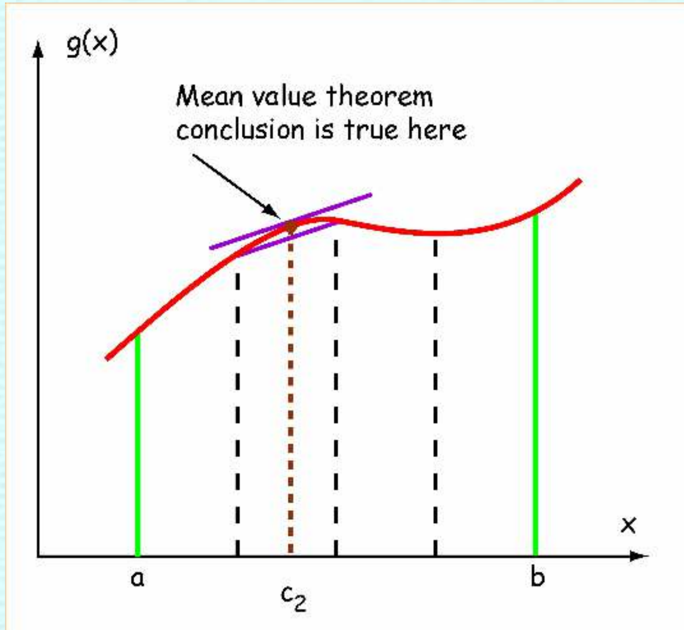
- Now, without the limits of integration, the indefinite integral is a function $g(x)$.

$$g(x) = \int f(x)\,dx$$

To the summit

- By the definition of the indefinite integral, $g'(x) = f(x)$.

- Since $g(x)$ is differentiable, the mean value theorem applies to it on [a,b] or on any subinterval of [a,b]. Let's apply it.



To the summit

- From the mean value theorem then:

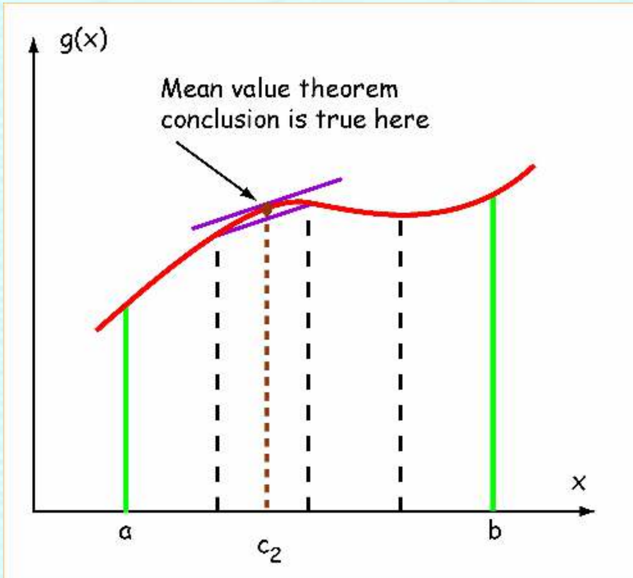
$$g'(c_1) = \frac{g(x_1) - g(a)}{\Delta x}$$

$$g'(c_2) = \frac{g(x_2) - g(x_1)}{\Delta x}$$

$$g'(c_3) = \frac{g(x_3) - g(x_2)}{\Delta x}$$

$$\ldots$$

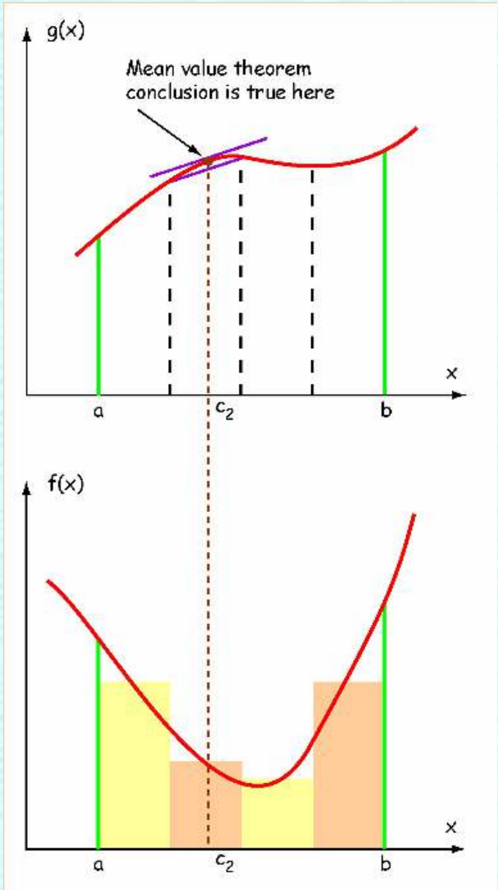
$$g'(c_n) = \frac{g(b) - g(x_{n-1})}{\Delta x}$$



To the summit

- So far we haven't done anything new.
  Now we will.

- Let's use the c_k from the mean value theorem as sample points for a Riemann sum of the original definite integral.



To the summit

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_k)\,\Delta x_k$$

- However $g'(x) = f(x)$ for all x by definition of an indefinite integral

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} g'(c_k)\,\Delta x_k$$

- Let's be explicitly write out the sum …

To the summit

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} [g'(c_1)\,\Delta x +$$

$$g'(c_2)\,\Delta x +$$

$$g'(c_3)\,\Delta x +$$

$$\ldots +$$

$$g'(c_n)\,\Delta x]$$

- Replace $g'(c_1)$ with its mean value thereom counterpart $[g(x_1) - g(a)]/\Delta x$, cancel the Δx, and repeat this operation for each term in the sum …

To the summit

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} [g(x_1) - g(a) +$$

$$g(x_2) - g(x_1) +$$

$$g(x_3) - g(x_2) +$$

$$\ldots +$$

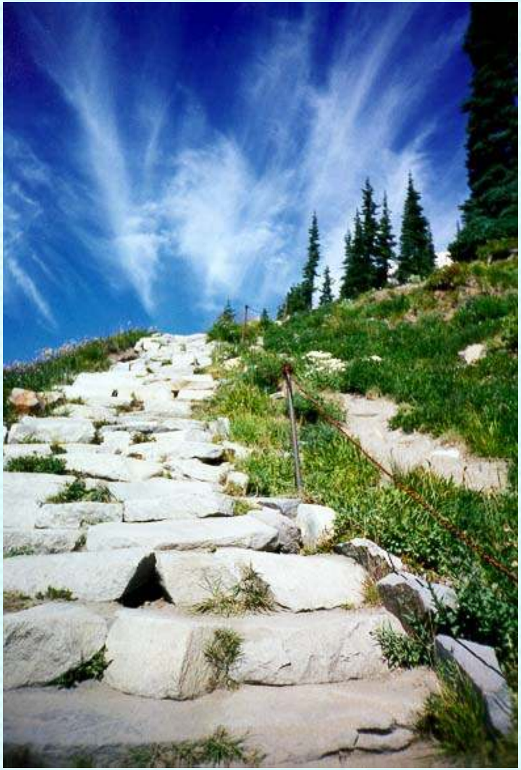
$$g(x_{n-1}) - g(x_{n-2}) +$$

$$g(b) - g(x_{n-1})]$$

- The only terms that survive the sum are the -$g(a)$ from the first term and the $g(b)$ from the last term!

The theorem

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0}\left[g(b) - g(a)\right] = g(b) - g(a)$$



- The quantity $g(b) - g(a)$ is the exact value of the definite integral, and is independent of the number of increments.

The theorem

- The definite integral can be calculated by evaluating the antiderivative at the upper limit of integration and then subtracting from it the value of the antiderivative at the lower limit of integration.

- This is what the fundamental theorem of calculus says.

Denali

The fundamental theorem of calculus

- If f(x) is an integrable function, and if $g(x) = \int f(x)\,dx$, then

$$\int_a^b f(x)\,dx = g(b) - g(a)$$

- The importance and utility of this theorem can hardly be overstated.

Example

- Evaluate $\int_1^4 x^2 dx$

$$\int_1^4 x^2 dx$$

$$= \frac{1}{3}x^3 \Big|_1^4$$

Read as "$1/3x^3$ evaluated from x = 1 to x = 4".

$$= \frac{1}{3}\cdot 4^3 - \frac{1}{3}\cdot 1^3$$

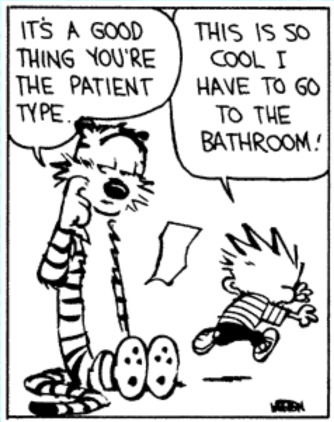
$$= 21$$

Notice how much easier this exact answer is than dividing the region under the graph into strips, and doing a bunch of algebra, only to get an approximate value.

Properties

- The following are some properties of definite integrals that are useful for both evaluating them and for understanding what they mean.

$$\int_a^b f(x)\,dx = g(b) - g(a)$$



Properties

- Integral with a negative integrand: $\int_1^4 (x^2 - 5x + 2)\,dx$

$$\int_1^4 (x^2 - 5x + 2)\,dx$$

$$= \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 2x\right)\Bigg|_1^4$$

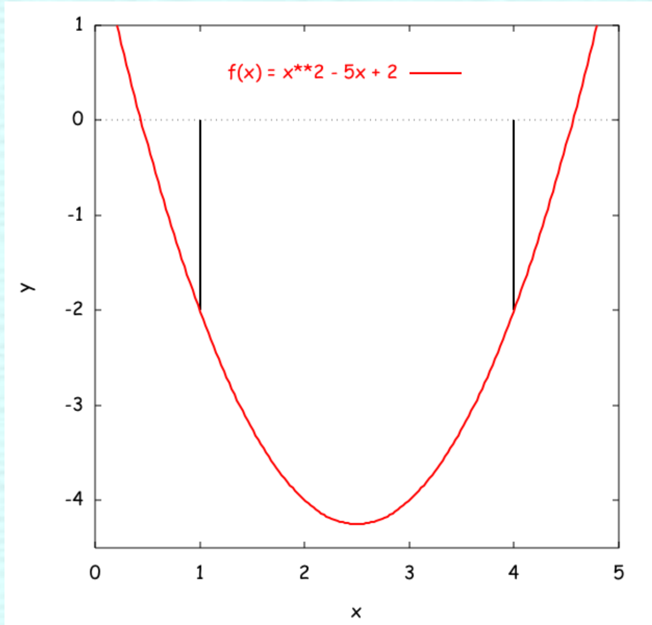
$$= \left(\frac{64}{3} - 40 + 8\right) - \left(\frac{1}{3} - \frac{5}{2} + 2\right)$$

$$= -10.5$$

How could an area be negative?!

Properties

- A plot of the integrand reveals the reason why. The region lies below the x-axis.



- The Riemann sum has the form $\sum f(x)\,\Delta x$. Each $f(x)$ is negative, and each Δx is positive. Every term in the sum is negative, so the definite integral is negative.

Properties

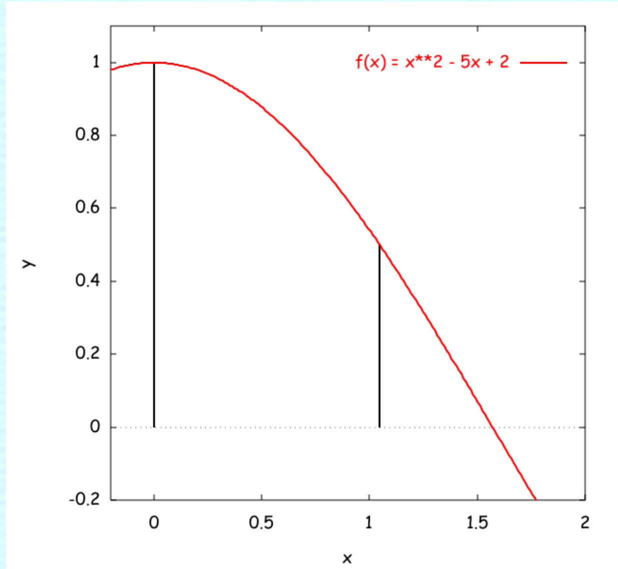
- Integral from a higher number to a lower number:

$$\int_{\pi/3}^{0} \cos x \, dx$$

$$= \sin x \Big|_{\pi/3}^{0}$$

$$= \sin(0) - \sin(\pi/3)$$

$$= 0 - \frac{\sqrt{3}}{2} = -0.86602$$



- The integral is negative, yet the integrand is positive. In this case each Δx is negative, since $\Delta x = (b - a)/n$, Δx will be negative whenever b is less than a.

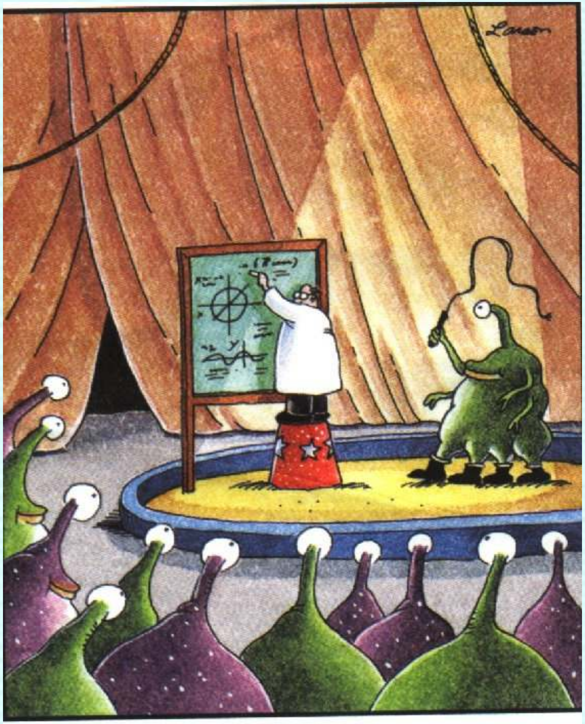
Properties

- Combining the previous observations, leads one to conclude that if f(x) and Δx are negative, the integral is positive.

$$\int_3^1 -x \, dx \quad = -\frac{1}{2}x^2\Big|_3^1$$

$$= -\frac{1}{2} + \frac{9}{2} = 4$$

Interlude



Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.

Properties

- Suppose you integrate $x^2 + 1$ from $x = 1$ to $x = 4$, and then integrate it from $x = 4$ to $x = 5$.

- The sum of the two areas equals the area of the region from $x = 1$ all the way to $x = 5$.

- This suggests that the two integrals should add up to the integral of $x^2 + 1$ from $x = 1$ to $x = 5$.

Properties

$$\int_1^4 \left(x^2+1\right) dx = \left(\frac{1}{3}x^3 + x\right)\Bigg|_1^4 = \frac{64}{3} + 4 - \frac{1}{3} - 1 = 24$$

$$\int_4^5 \left(x^2+1\right) dx = \left(\frac{1}{3}x^3 + x\right)\Bigg|_4^5 = \frac{125}{3} + 5 - \frac{64}{3} - 4 = 21\frac{1}{3}$$

$$\int_1^5 \left(x^2+1\right) dx = \left(\frac{1}{3}x^3 + x\right)\Bigg|_1^5 = \frac{125}{3} + 5 - \frac{1}{3} - 1 = 45\frac{1}{3},$$

which equals $24 + 21\frac{1}{3}$

- In general, $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Properties

- We've seen that the indefinite integral of a sum of two functions equals the sum of the integrals, and that the integral of a constant times a function is the constant times the integral.

- By the fundamental theorem of calculus, these properties hold for definite integrals too.

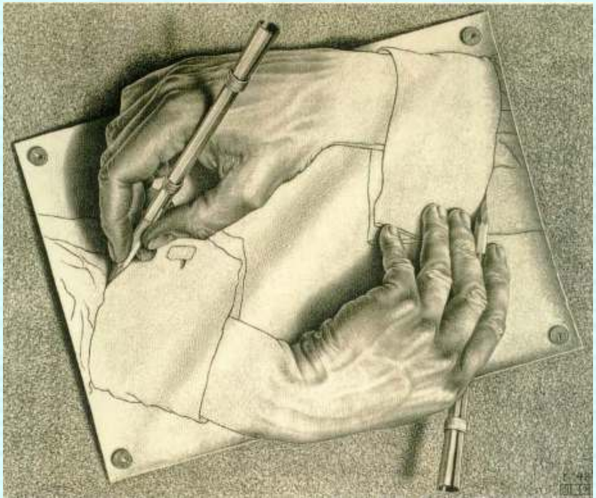
$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

$$\int_a^b k\, f(x)\, dx = k \int_a^b f(x)\, dx$$

Properties

- Integrals between symmetric limits: $$\int_{-a}^{a} f(x)\,dx$$

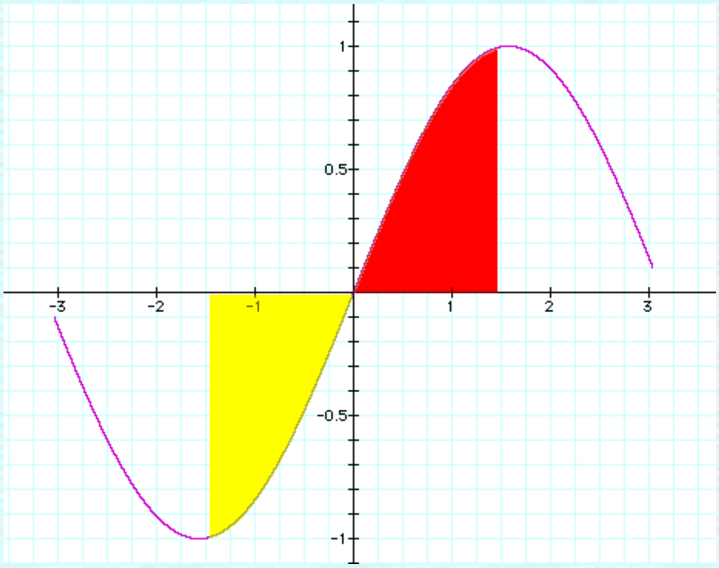
- If f(x) happens to be an odd function (such as x^5 or sin x) or an even function (such as x^6 or cos x) then the integral has properties that make it easier to evaluate.



Drawing Hands
M.C. Escher, 1948

Properties

- An odd function is one where f(-x) = -f(x), like sin(x).

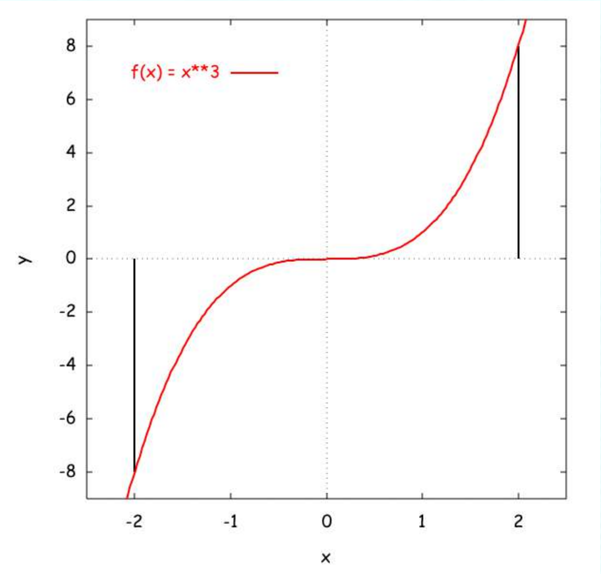


- The area of the region from x = -a to 0 equals the area from x = 0 to a, but the signs of the integrals will be opposite.
- Thus, the integral is zero!

Properties

$$\int_{-2}^{2} x^3 \, dx = \frac{1}{4} x^4 \Big|_{-2}^{2}$$

$$= \frac{1}{4}\left[2^4 - (-2)^4\right] = 0$$



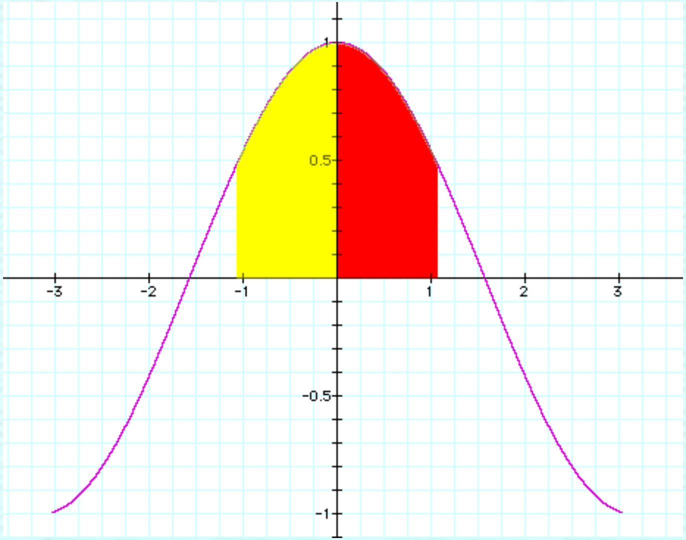
- If you have an integral of an odd function over a symmetric interval, you don't have to explicitly evaluate it. Simply write "equals 0, odd function over symmetric interval".

Properties

- An even function is one where f(-x) = f(x), like cos(x).

- The areas of the regions from x = -a to 0 and from x = 0 to a, are again the same. This time the integrals have the same sign.

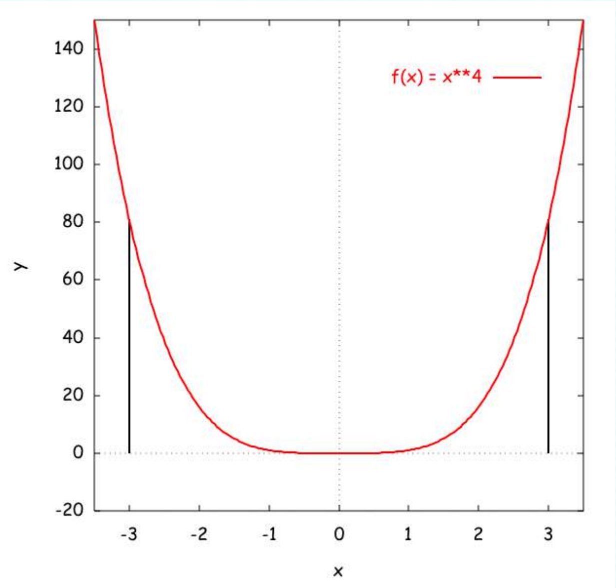
- Thus, you can integrate from 0 to a and then double the result.



Properties

$$\int_{-3}^{3} x^4 \, dx = \frac{1}{5} x^5 \Big|_{-3}^{3} = \frac{1}{5}\left[3^5 - (-3)^5\right] = 97.2$$

$$2\int_{0}^{3} x^4 \, dx = \frac{2}{5} x^5 \Big|_{0}^{3} = \frac{2}{5}\left[3^5 - 0\right] = 97.2$$



- If you have an integral of an even function over a symmetric interval, it's usually easier to integrate over half the interval (because of the zero) and double it.

Playtime

- During your in-class problem solving session today you'll evaluate some definite integrals using the fundamental theorem of calculus.