Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into mean and extreme ratios, that is, the Golden Mean. The first may be compared to a measure of gold, the second to a precious jewel.

Johannes Kepler

School of the Art Institute of Chicago

Geometry of Art and Nature

Frank Timmes

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flash.uchicago.edu/~fxt/class_pages/class_geom.shtml

Syllabus

1	Sept 03	Basics and Celtic Knots
2	Sept 10	Golden Ratio
3	Sept 17	Fibonacci and Phyllotaxis
4	Sept 24	Regular and Semiregular tilings
5	Oct 01	Irregular tilings
6	Oct 08	Rosette and Frieze groups
7	Oct 15	Wallpaper groups
8	Oct 22	Platonic solids
9	Oct 29	Archimedian solids
10	Nov 05	Non-Euclidean geometries
11	Nov 12	Bubbles
12	Dec 03	Fractals

Sites of the Week

- ccins.camosun.bc.ca/~jbritton/goldslide/jbgoldslide.htm
- www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/phi.html
- www.math.nus.edu.sg/aslaksen/teaching/math-art-arch.shtml

Class #2

- Ruler and compass fun
- The golden ratio
- Golden polygons
- Golden ratios in art

Greek culture

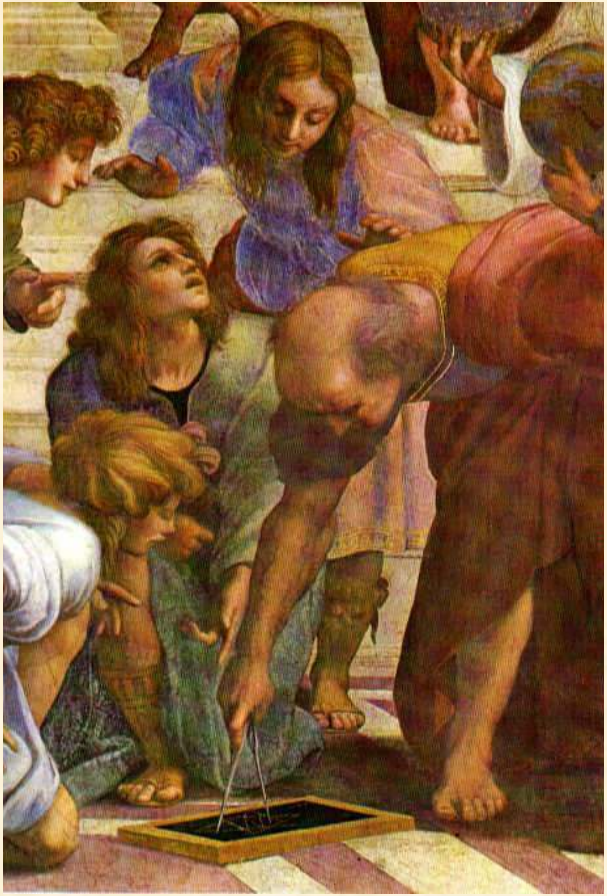
- In Alexandria, a center of Greek culture, there arose a state-supported library dedicated to the study of the muses.

School of Athens
1510, Raphael Sanzio

Euclidean geometry

- Euclid, the Greek mathematician of about 300 BC, wrote The Elements, which is a collection of 13 books on geometry.

- It's one of the most celebrated works of the human mind.

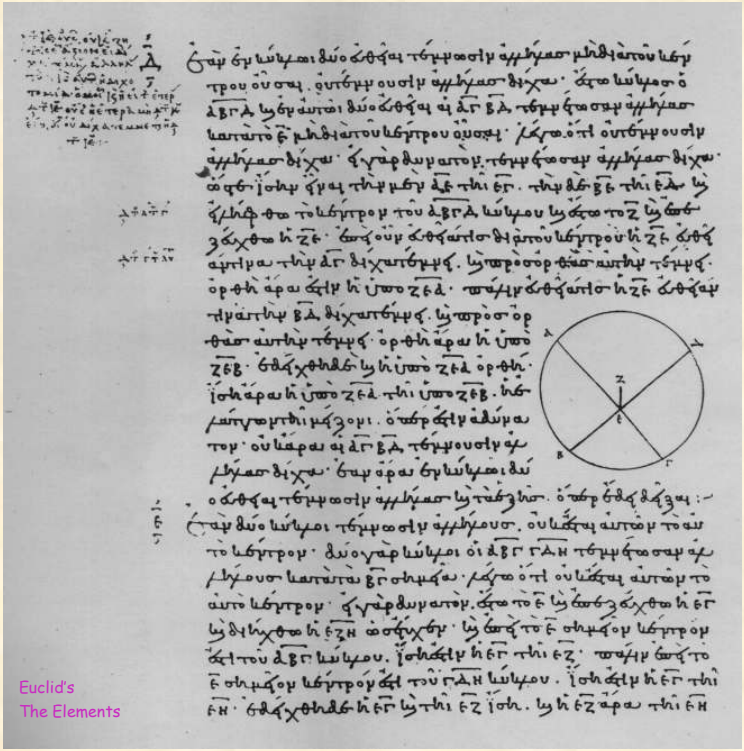


Euclid in Raphael's School of Athens

Euclidean geometry

- It starts from basic definitions called axioms (self-evident starting points).

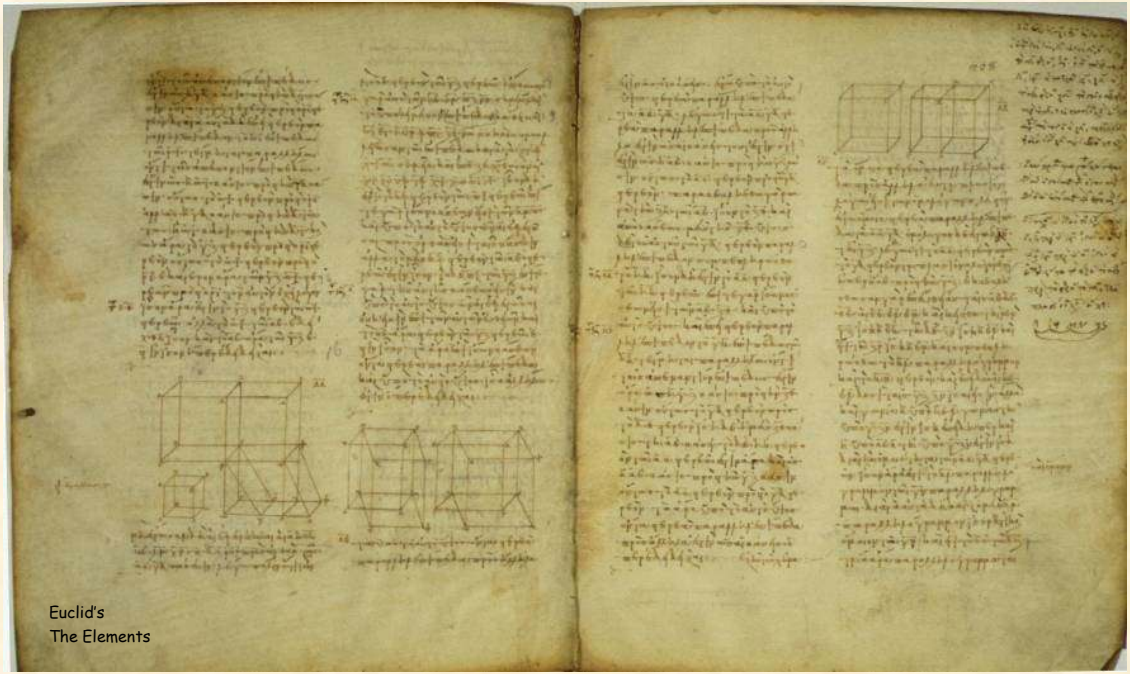
- An example is the first axiom:
  Through any two distinct points, it is possible to draw exactly one straight line.



Euclid's
The Elements

Euclidean geometry

- From these Euclid develops results (called propositions) about geometry which he proves (using formal logic) using only the axioms and previously proved propositions.



Euclid's
The Elements

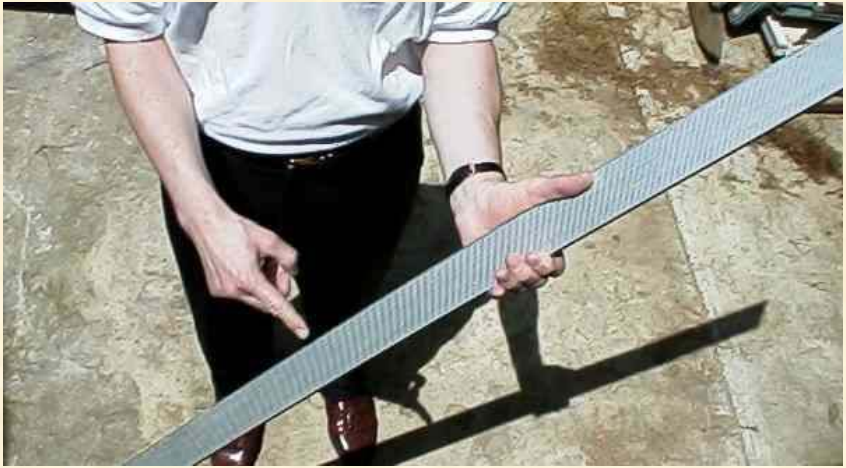
Euclidean games

- Euclid seems to have regarded geometry as a sort of game:
  - we can use our straight-edge to join two points and extend the line as far as we like.
  - we can use our compass to draw a circle, centered at one of the points and passing through the other.

Euclidean games

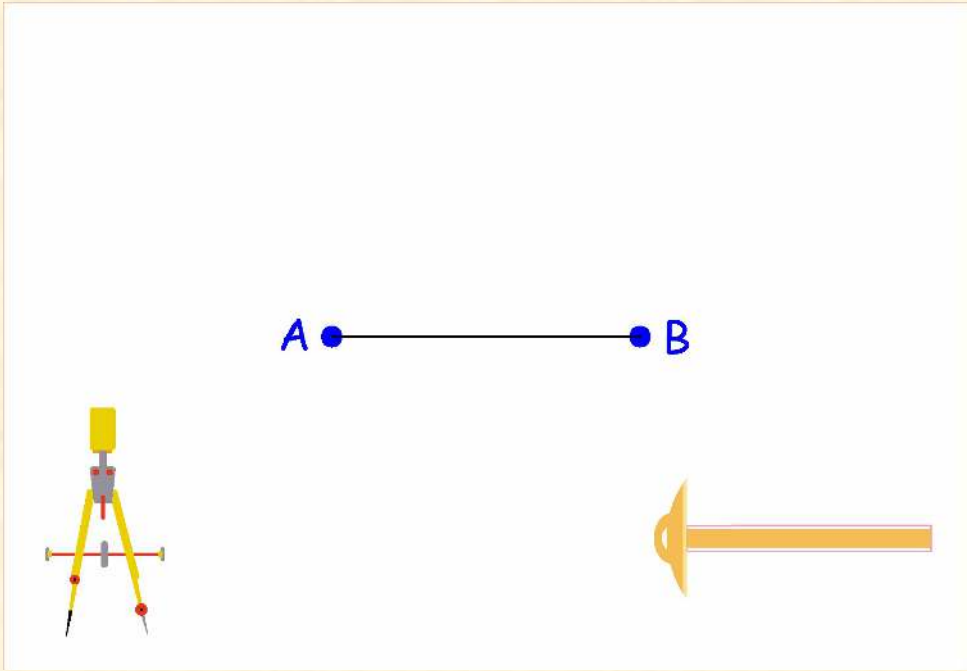
- Further points, found as the places where straight lines and circles meet, may be used to find other points, and so on.

- Euclid's rules don't allow you to have any marks on your ruler!



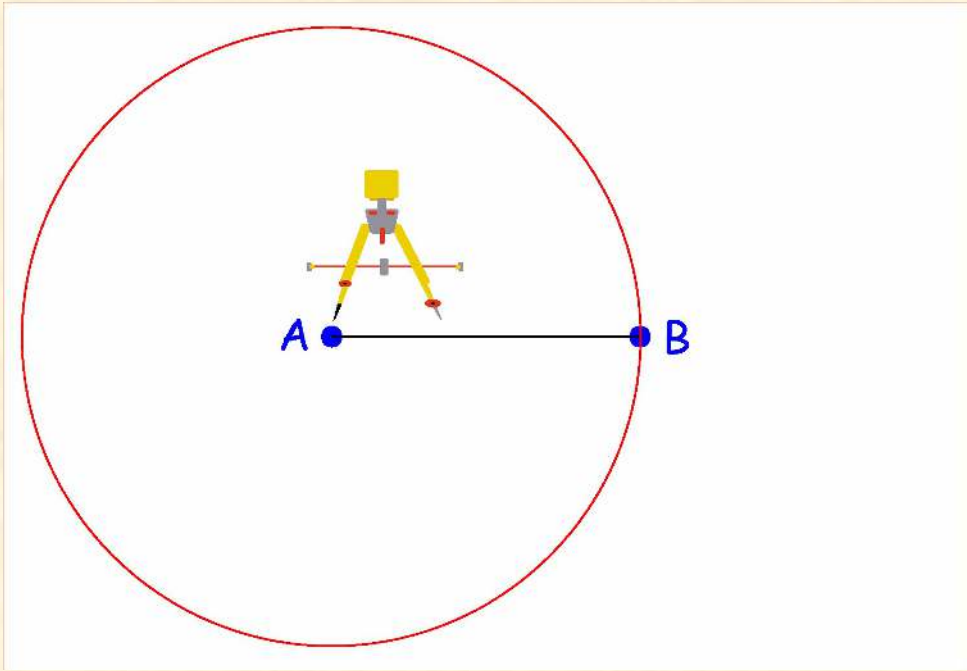
Euclidean games

- For instance, to find the exact center of any line AB:



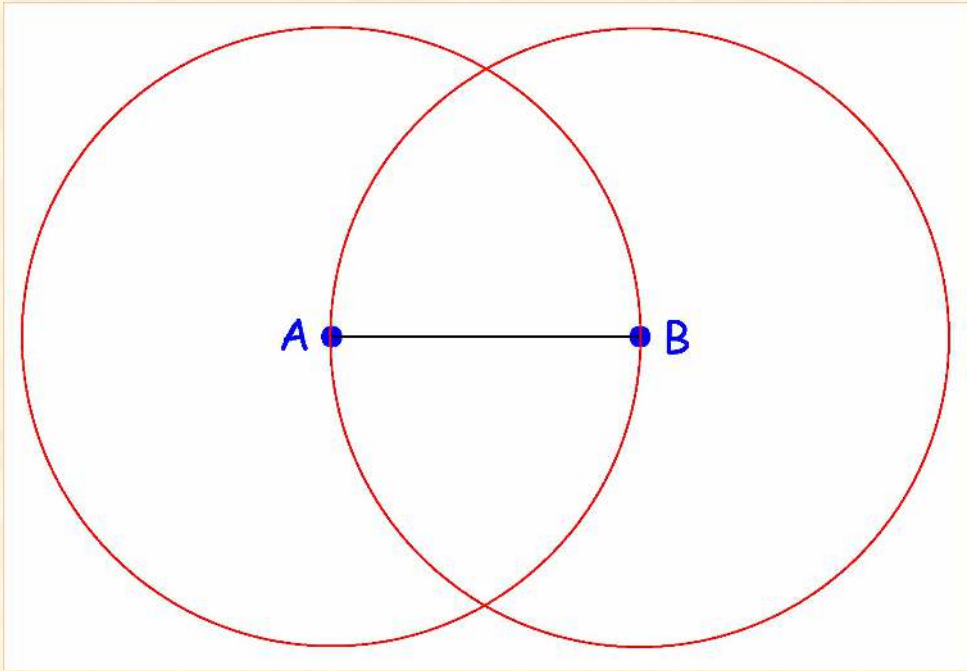
Euclidean games

- Put your compass point at point A, put the pencil at point B, and draw the circle.



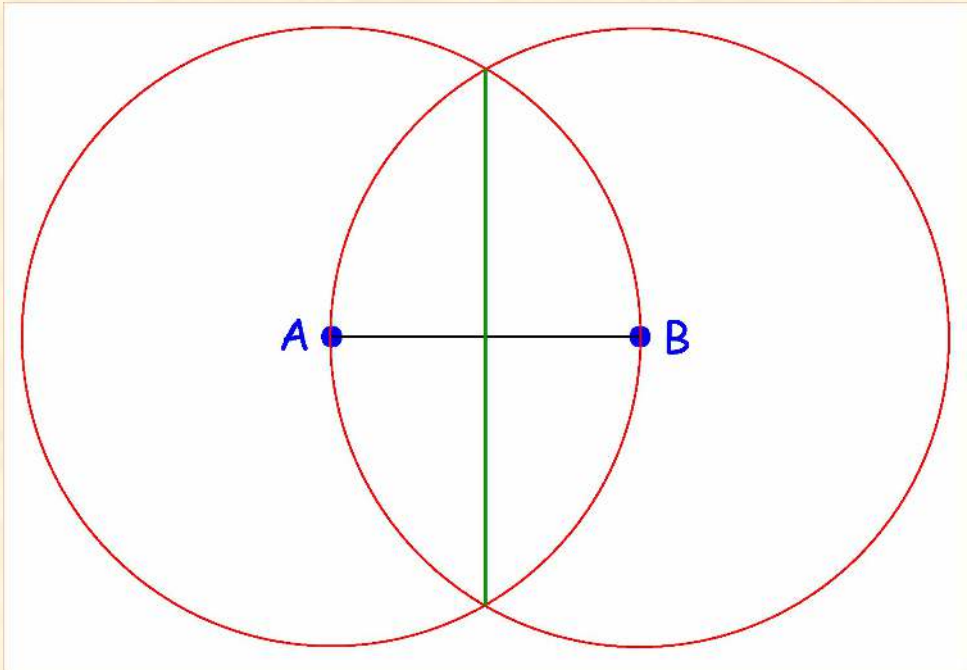
Euclidean games

- Draw another circle in the same way with center at point B.



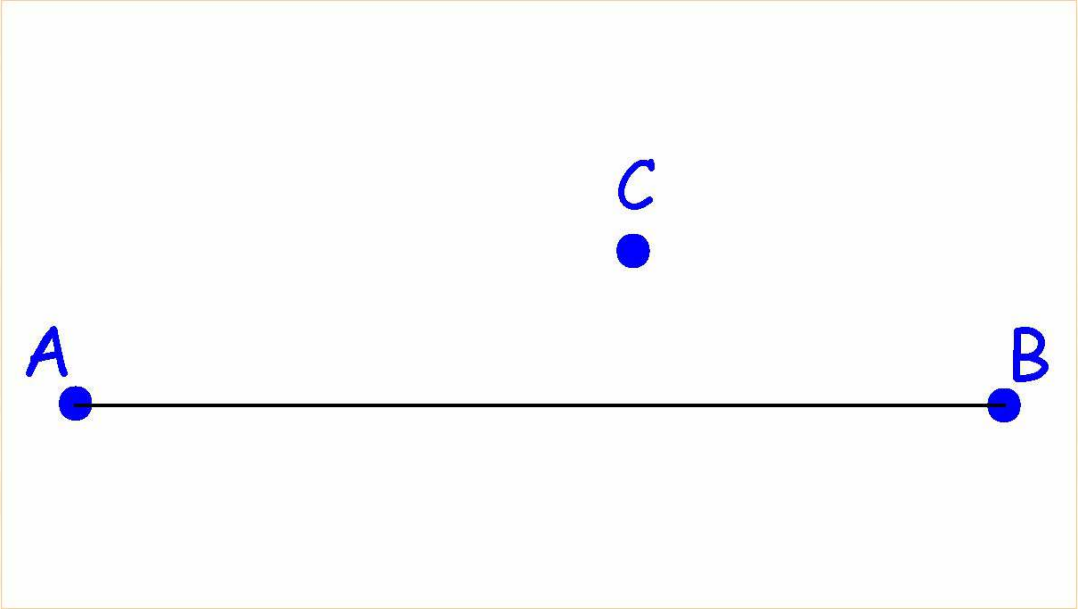
Euclidean games

- The two circles cross at two points. Join these points, and we have a straight line at 90° to the original line that goes exactly through its center.



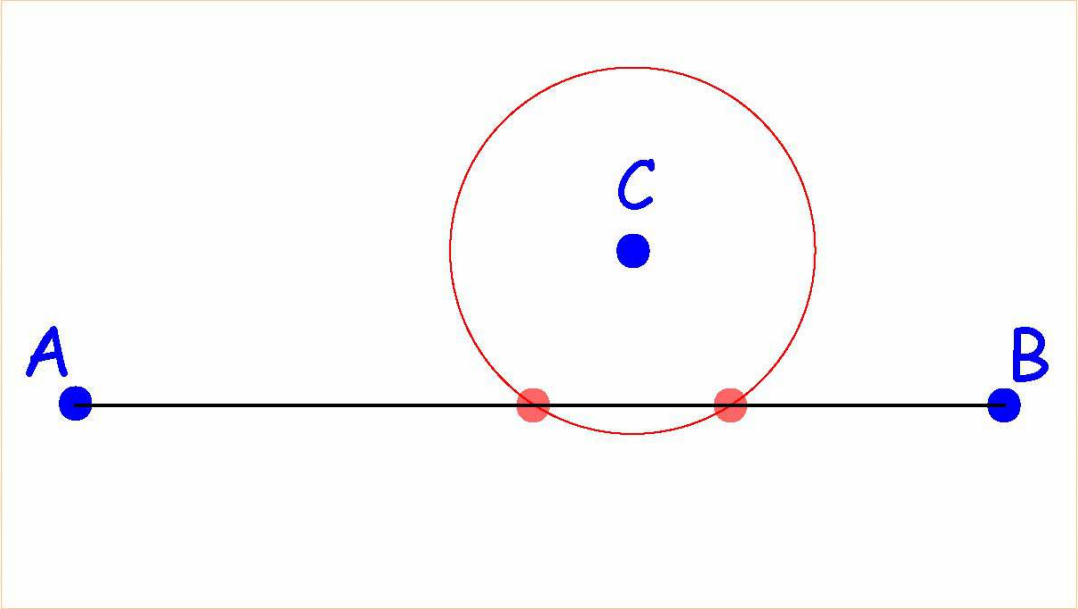
Euclidean games

- Let's do another. Given a line and a point not on the line, construct a perpendicular line from the point to the line.



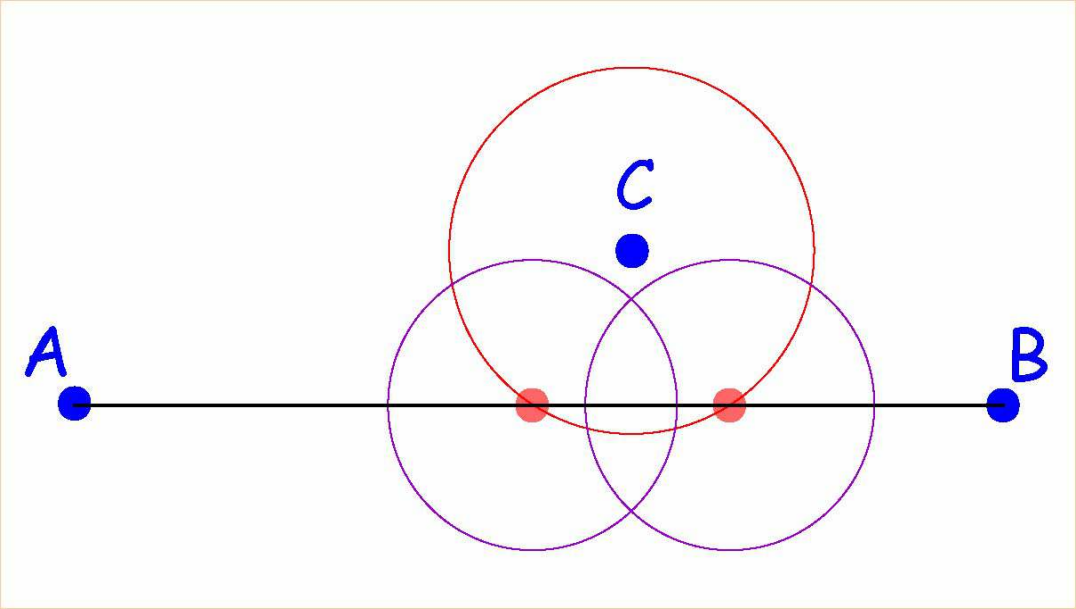
Euclidean games

- Put your compass point at point C, and draw any circle such that it intersects the line at two points.



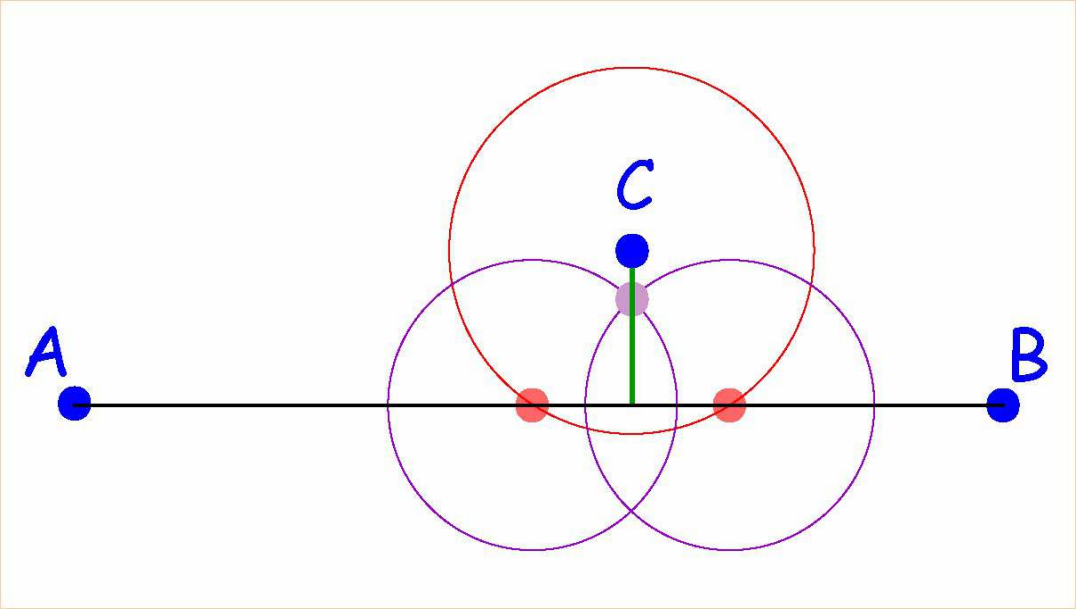
Euclidean games

- At each intersecting point draw a circle, such that the two circles overlap.
  This should look familiar …



Euclidean games

- Drawing a line from point C through the point where the two purple circles intersect to the original line. This new green line is exactly 90° from the original.

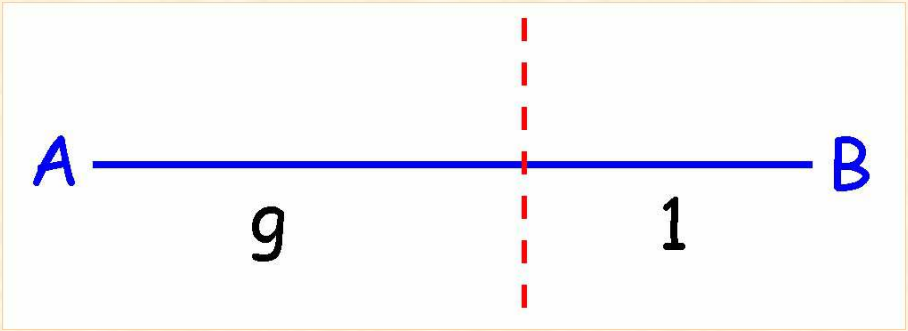


Golden Ratio

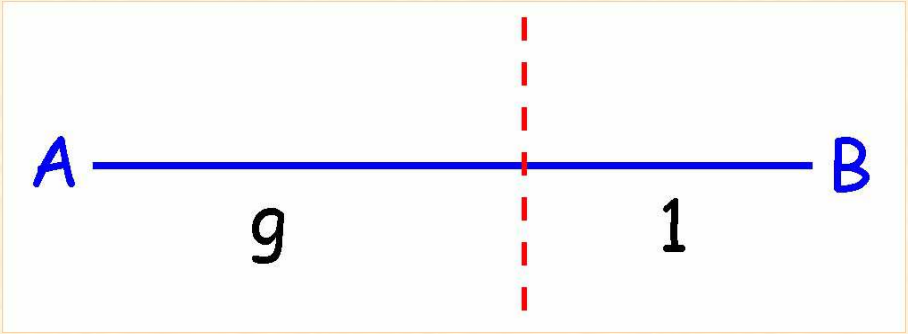
- In Book 6, Proposition 30, Euclid shows how to divide a line into its "mean and extreme ratio", which we'll simply call the "golden ratio".

- That is, divide a line into two parts such that:

$$\frac{\text{small}}{\text{large}} = \frac{\text{large}}{\text{total}}$$



Golden Ratio



$$\frac{\text{small}}{\text{large}} = \frac{\text{large}}{\text{total}} \qquad \frac{1}{g} = \frac{g}{1+g} \qquad g^2 = 1 + g$$

$$g^2 - g - 1 = 0$$

$$g = \frac{1 \pm \sqrt{5}}{2}$$

Golden Ratio

$$g = \frac{1 \pm \sqrt{5}}{2}$$

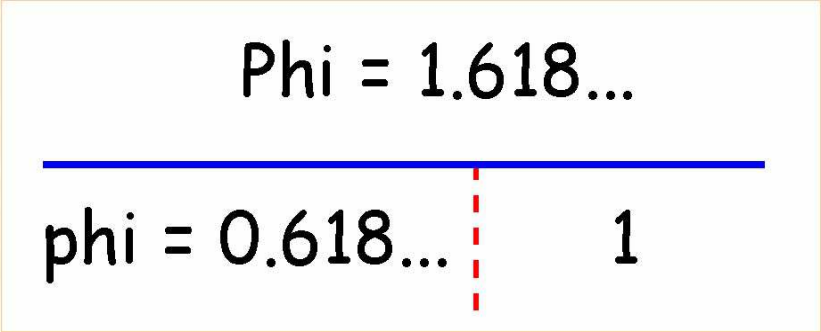
- We will name the positive square root solution Phi, and the negative square root solution phi, using the first letter to tell us if we want the bigger value or the smaller one.

$$\text{Phi} = \frac{1 + \sqrt{5}}{2} = 1.6180339887\text{K}$$

$$\text{phi} = \frac{1 - \sqrt{5}}{2} = 0.6180339887\text{K}$$

- Notice that their decimal parts are identical!

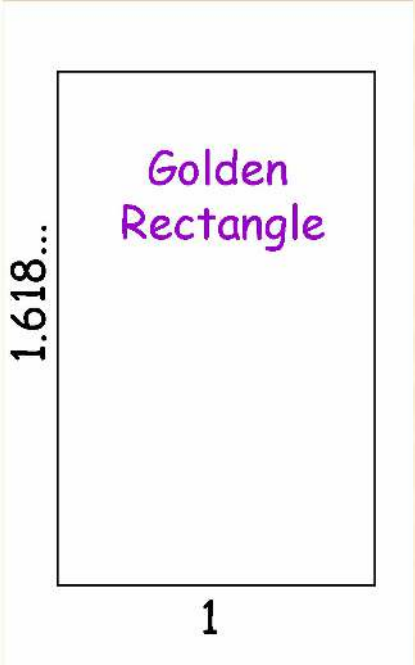
Golden Ratio

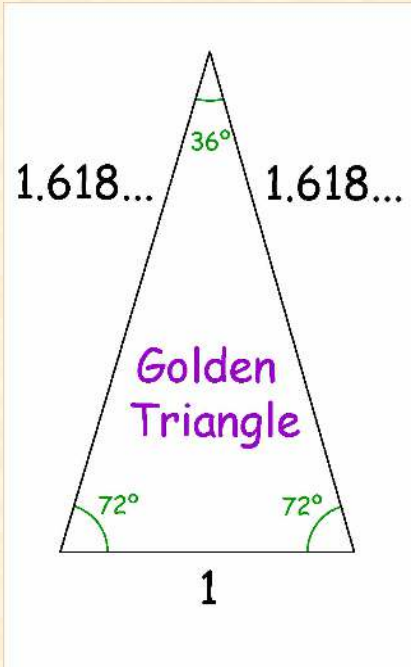


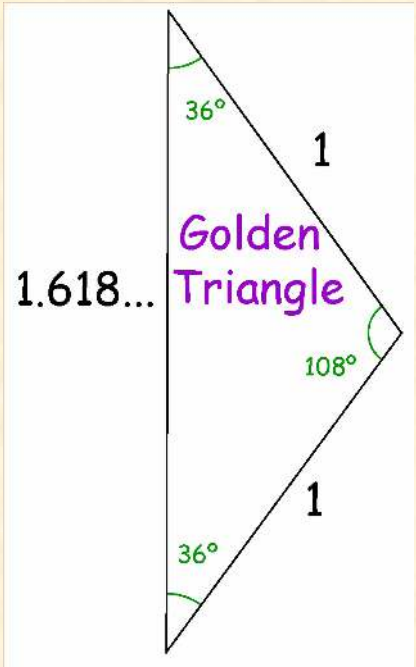
- Euclid called the 0.618… dividing point splitting a line in its "mean and extreme ratio".

- We'll call it the golden ratio, golden mean, or golden section.

Golden polygons

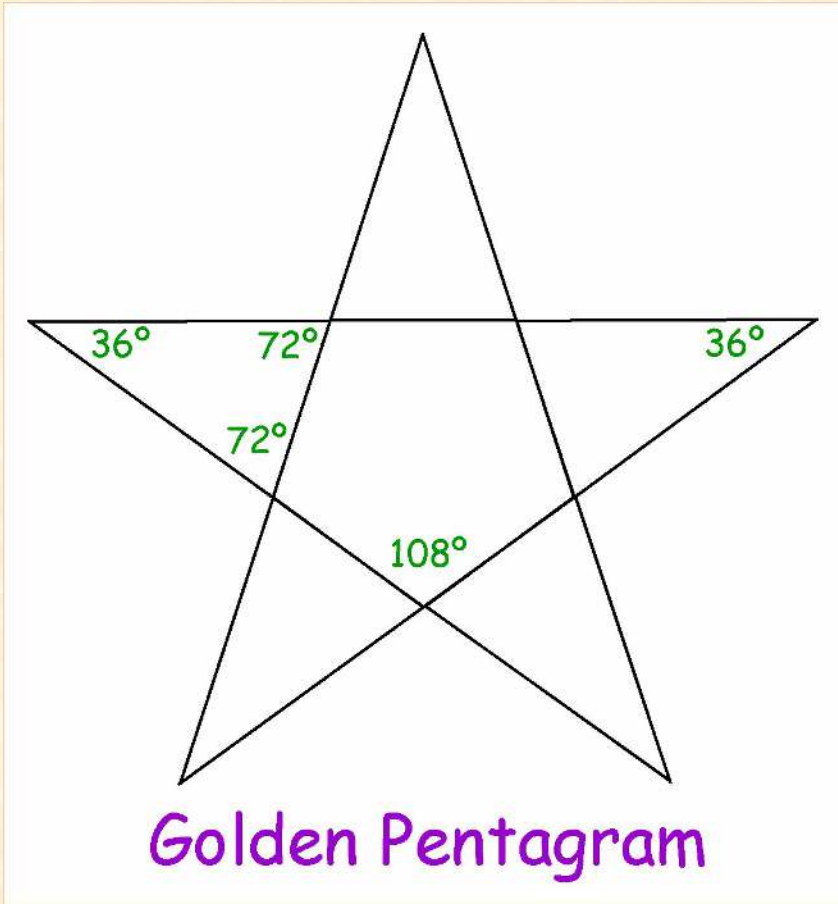






- The golden ratio is one of the most celebrated numbers in art, nature, and mathematics.

Golden polygons



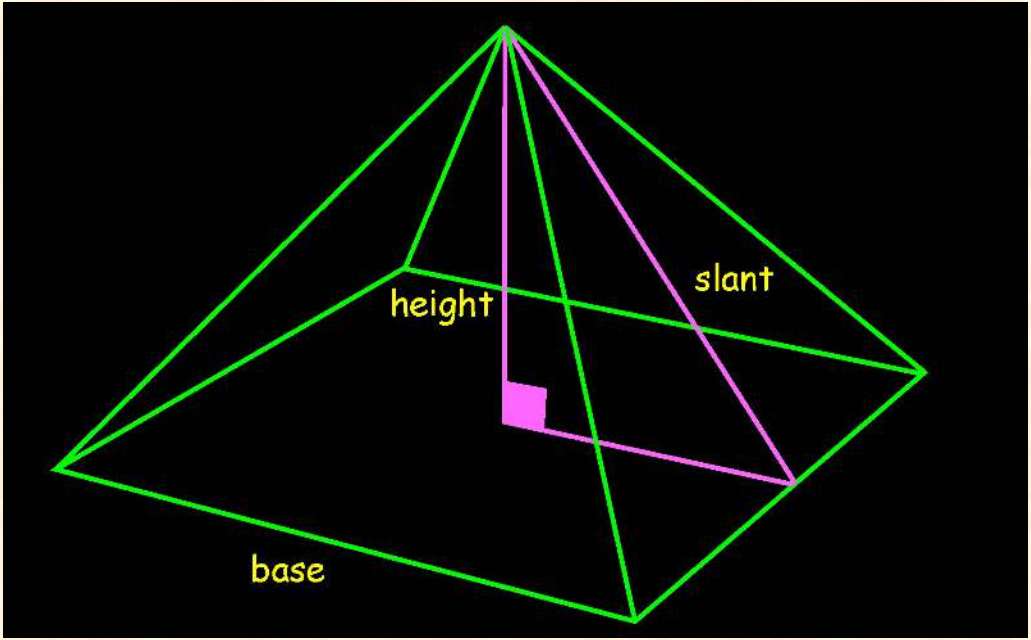
Egyptian pyramids

- The Great Pyramid supposedly contains the golden ratio in several aspects of its design. Let's see if it does.

Egyptian pyramids

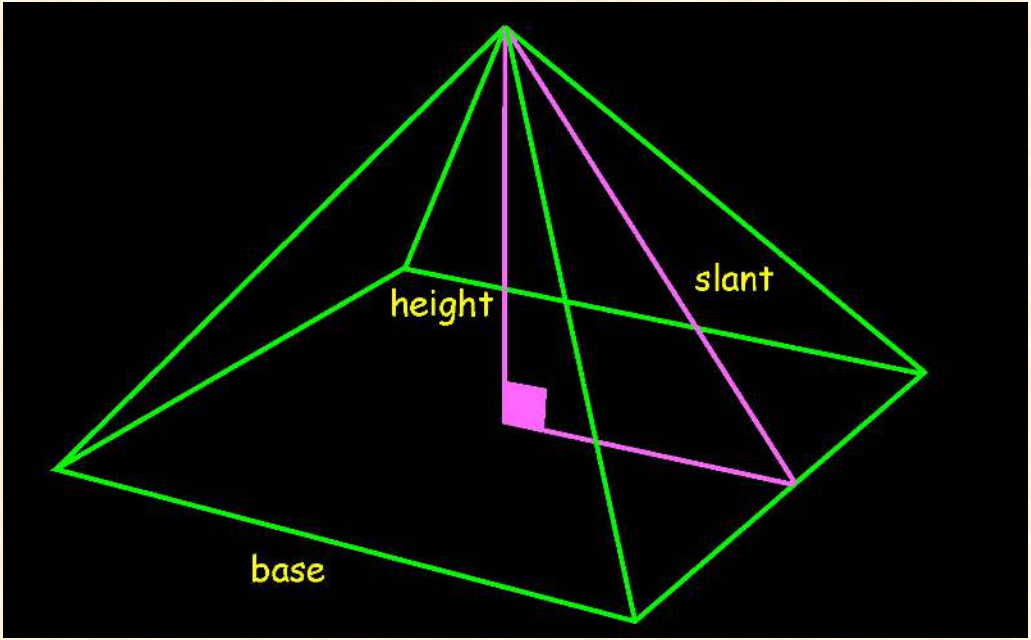
- If we take a cross-section through a pyramid we get a triangle.

- The dimensions of the Great Pyramid of Cheops, determined by various expeditions are: height = 146.515 m and base = 230.363 m.



Egyptian pyramids

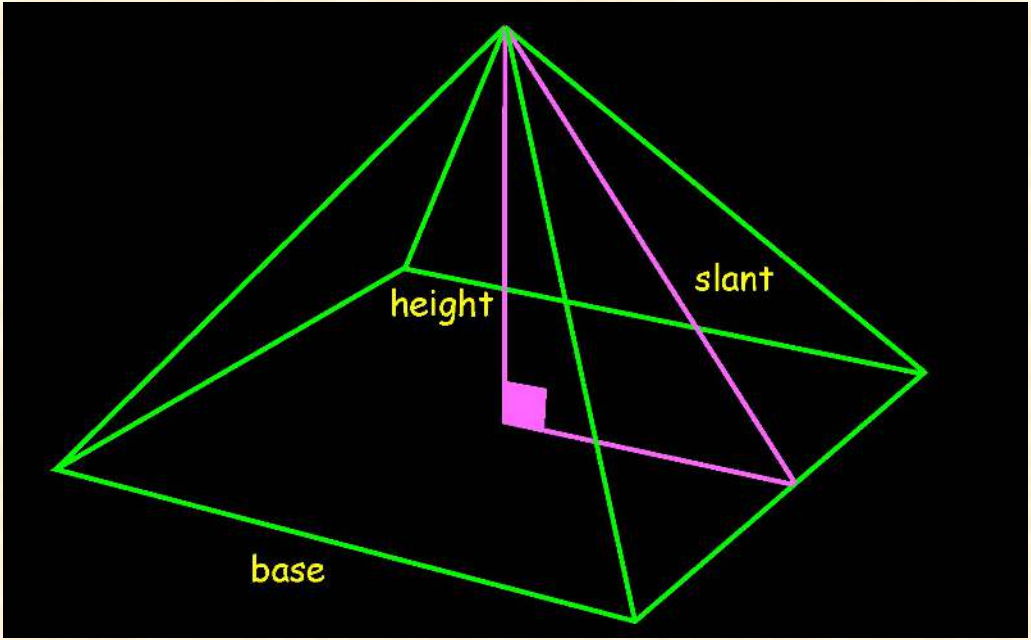
- Half the base is 0.5 • 230.363 = 115.182 m.
- So, $\text{Slant}^2 = 146.515^2 + 115.182^2$ Thus, slant = 186.369 m



Egyptian pyramids

- Dividing the slant by half the base gives 186.369/115.182 = 1.6804

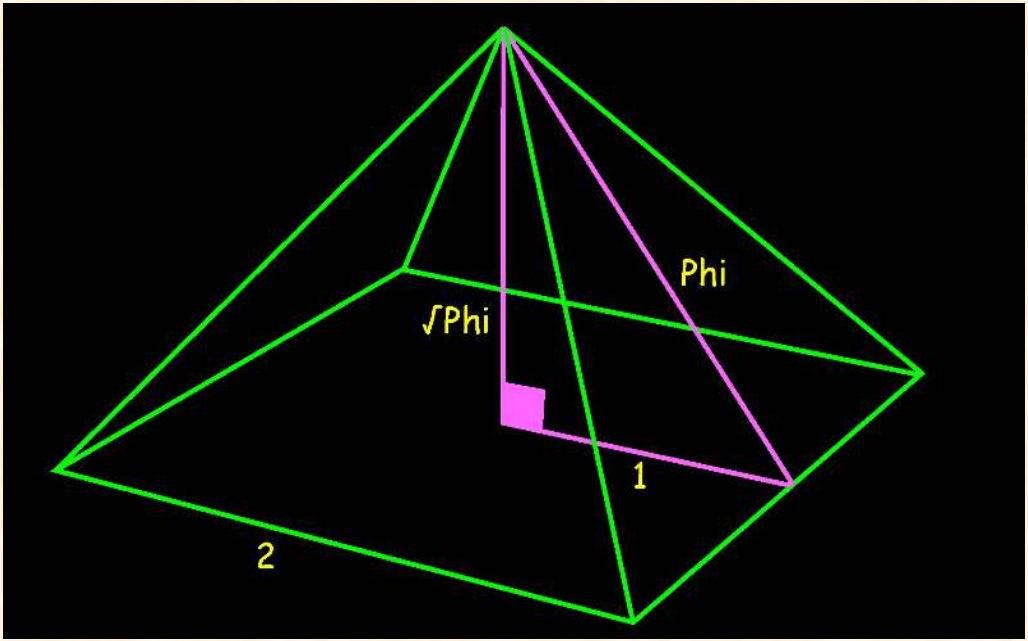
- Which differs from Phi (1.6803) by only one unit in the fifth decimal point!



Egyptian pyramids

- The "Egyptian triangle" has a base of 1 and a hypotenuse equal to Phi.
  Its height h, by the Pythagorean theorem, is then sqrt(Phi)

- Thus, the sides of the Egyptian triangle are in the ratio $1 : \sqrt{\text{Phi}} : \text{Phi}$



Pythagoras

- Pythagoras (560-480 BC), the Greek geometer, was especially interested in the golden section, and showed that it was a prevalent tendency for the proportions of the human body.

Pythagoras in Raphael's School of Athens

Parthenon

- Pythagoras' discoveries of the proportions of the human figure supposedly had a tremendous effect on Greek architecture.

Parthenon

- The Parthenon is perhaps the loudest claim of a mathematical approach to art.

Parthenon

- Once its ruined triangular pediment is restored, ...

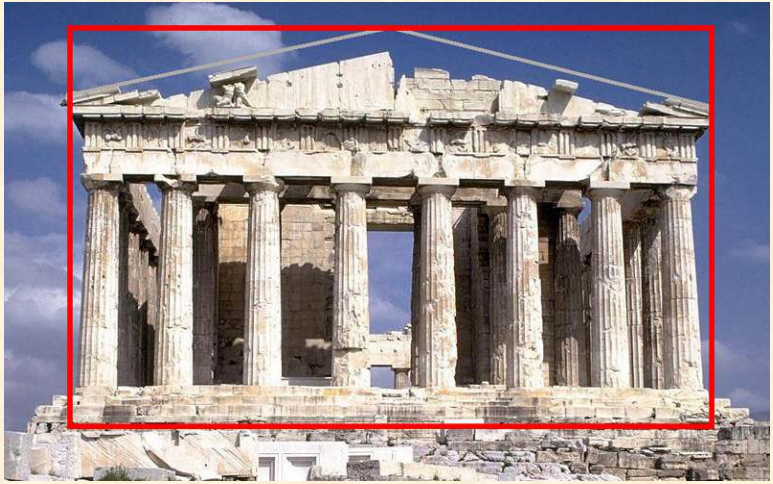
Parthenon

- …the ancient temple fits roughly within a golden rectangle.

Parthenon

- Are you skeptical?

- If you set about measuring a complicated structure, you will quickly have on hand a great abundance of lengths to play with. If you have sufficient patience to juggle them about in various ways, you are certain to come out with many figures which coincide with important historical dates or figures in the sciences.



Pacioli's rediscovery

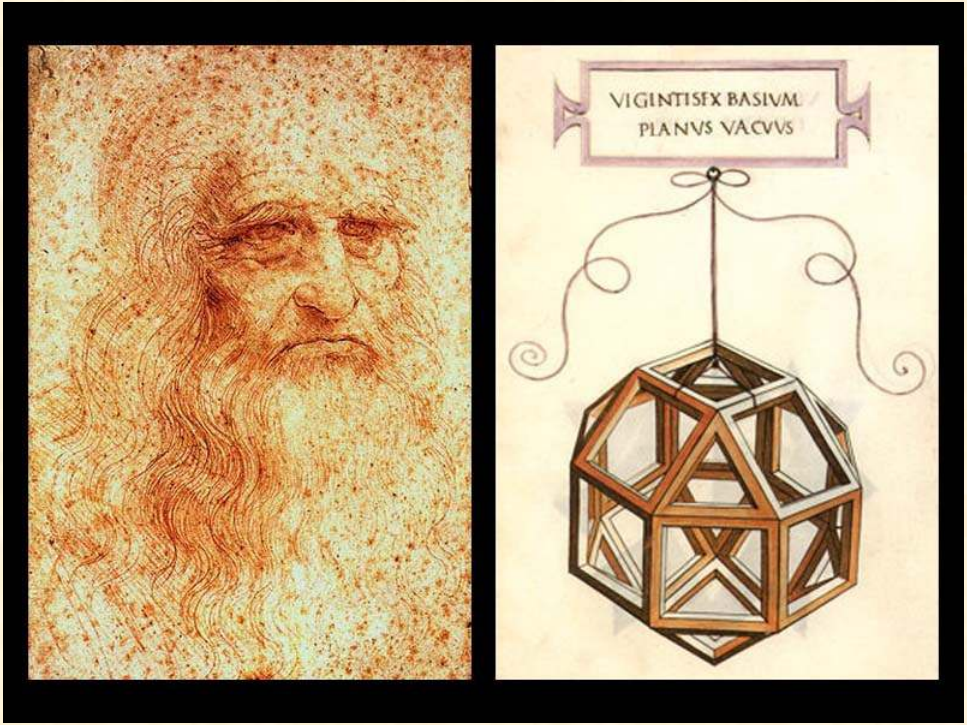
- In the 16th Century, Luca Pacioli (1445-1514), geometer and friend of the great Renaissance painters, rediscovered the "golden secret".

- His publication devoted to the number phi, Divina Proportione, was illustrated by no less an artist than ...

Fra' Luca Pacioli (attributed to Jacopo de Barbari)

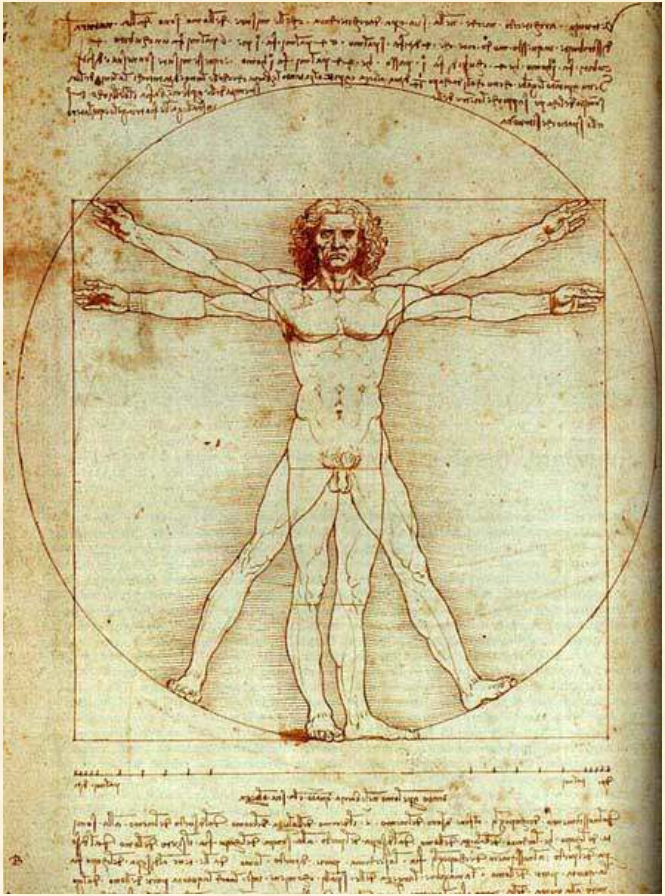
Da Vinci

- …Leonardo da Vinci (1451-1519). Leonardo had for a long time displayed an ardent interest in the mathematics of art and nature.



Da Vinci

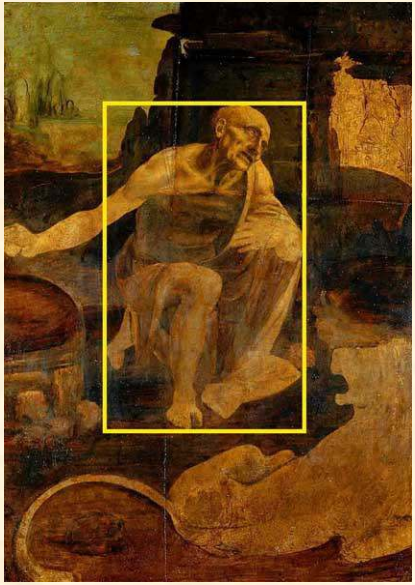
- He had earlier, like Pythagoras, made a close study of the human figure and had noted how all its different parts were loosely related by the golden section.

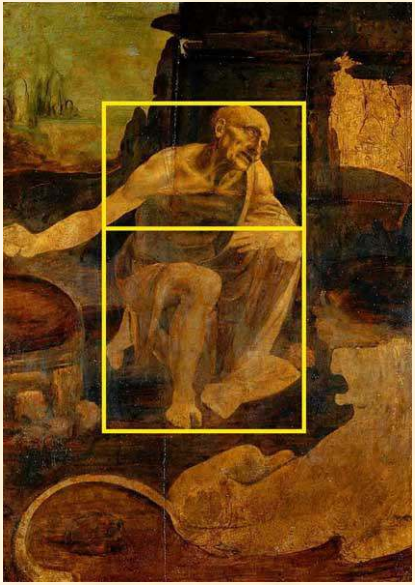


Study of Human Proportions
According to Vitruvious

Da Vinci

- Leonardo's unfinished canvas of Saint Jerome shows the great scholar with a lion lying at his feet.





- A golden rectangle fits so neatly around the central figure that it is often claimed the artist deliberately painted it that way. Are you skeptical?

Da Vinci

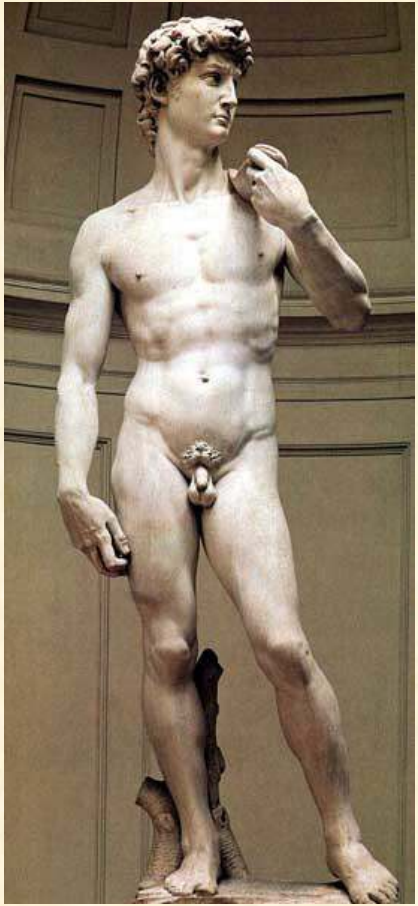
- Golden rectangles in Da Vinci's Mona Lisa seem to abound. Visit ccins.camosun.bc.ca/~jbritton/jbmona.htm to interactively add golden rectangles to this famous piece.

- Also harking back to classical themes for inspiration, Renaissance artists like Michelangelo and Raphael once more began to construct their compositions on the golden ratio.

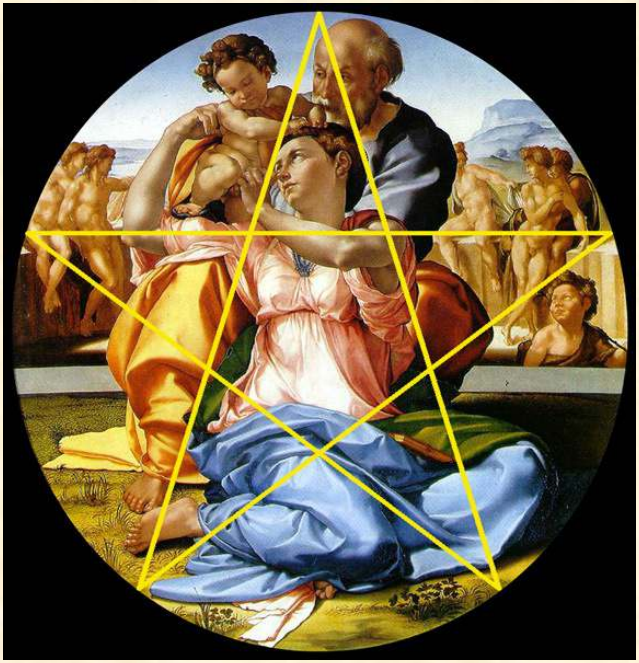
Golden David

- The proportions of Michelangelo's David conform to the golden ratio from the location of the navel with respect to the height, to the placement of the joints in the fingers.

- www.maths.adelaide.edu.au/pure/pscott/place/pm10/pm10.html has many details.



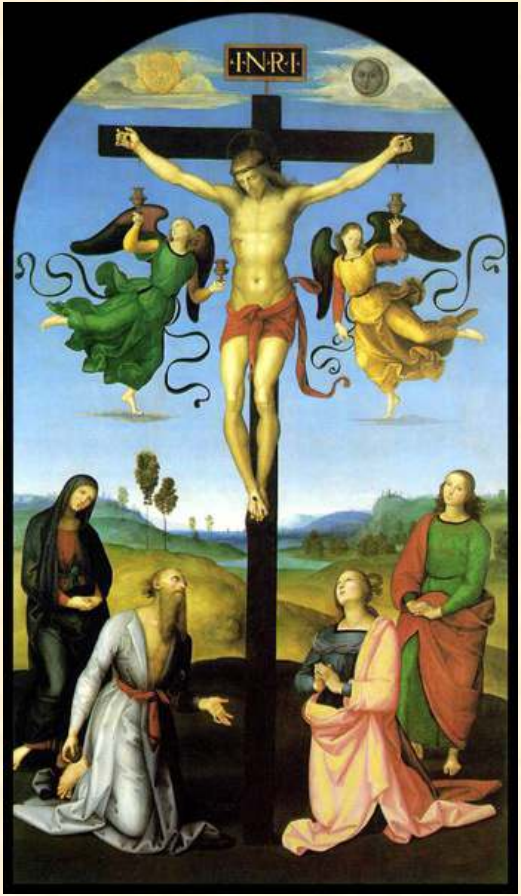
Renaissance



- Michelangelo's Holy Family is claimed notable for its positioning of the principal figures within a golden pentagram.

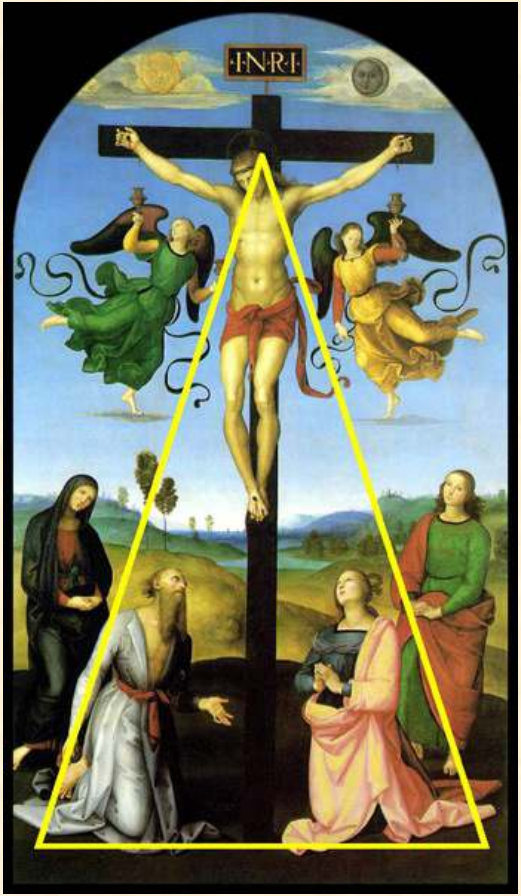
Renaissance

- Raphael's Crucifixion …



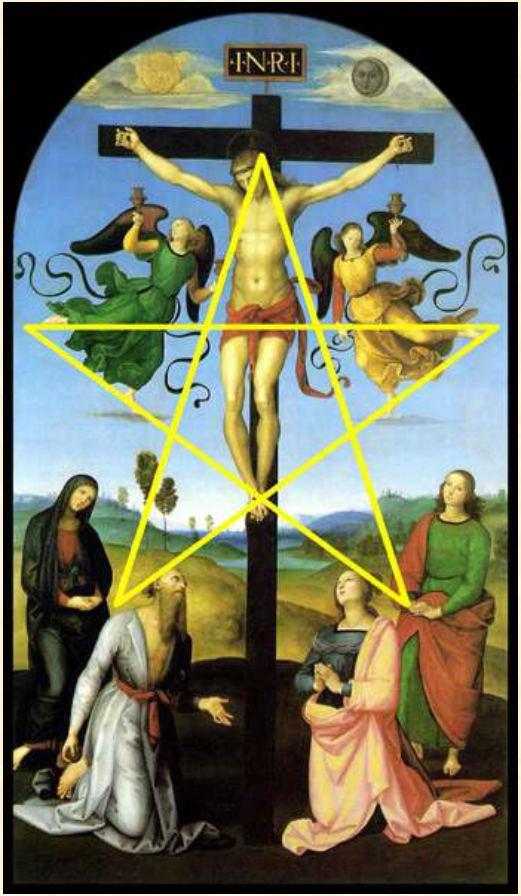
Renaissance

- ..is another claim. The principal figures seem to outline a golden triangle ...



Golden Ratio

- which can be used to locate one of its supposed pentagrams.

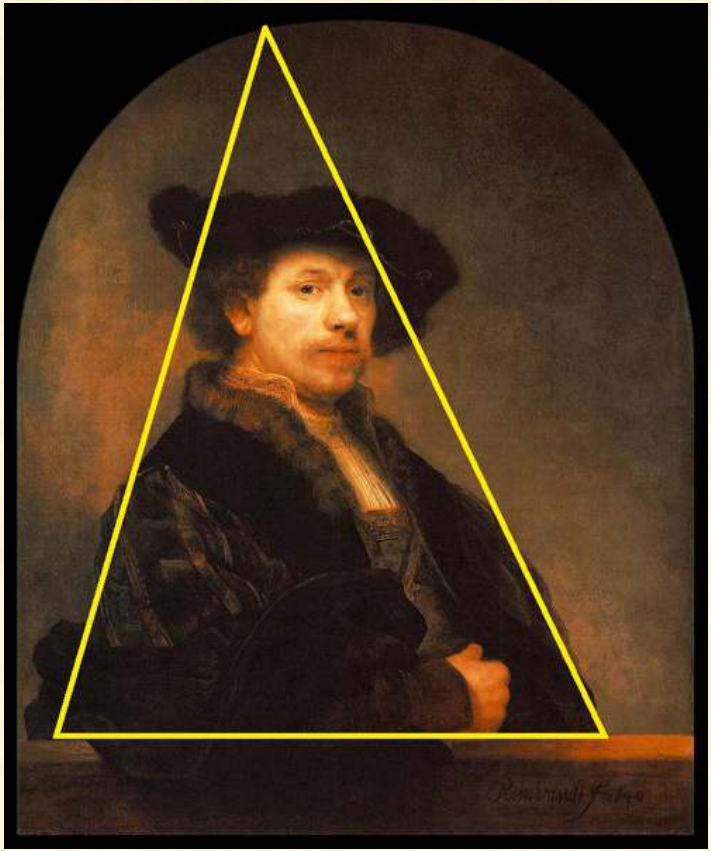


Van Rijn

- This self-portrait by Rembrandt (1606-1669) …

Van Rijn

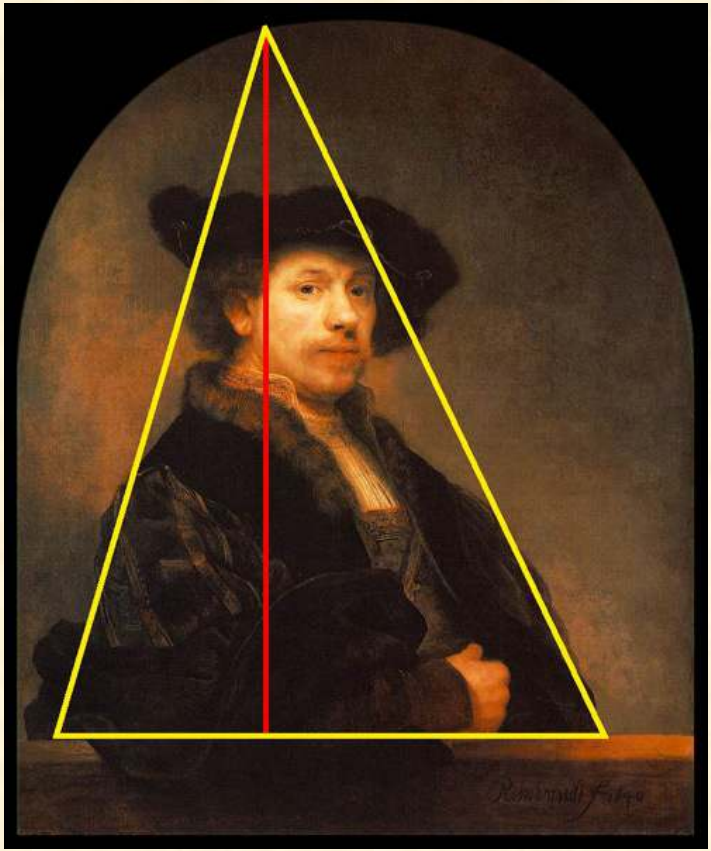
- …is an example of triangular composition - holding together an intricate subject within three straight lines. The different lengths of the sides add a little variety.



Van Rijn

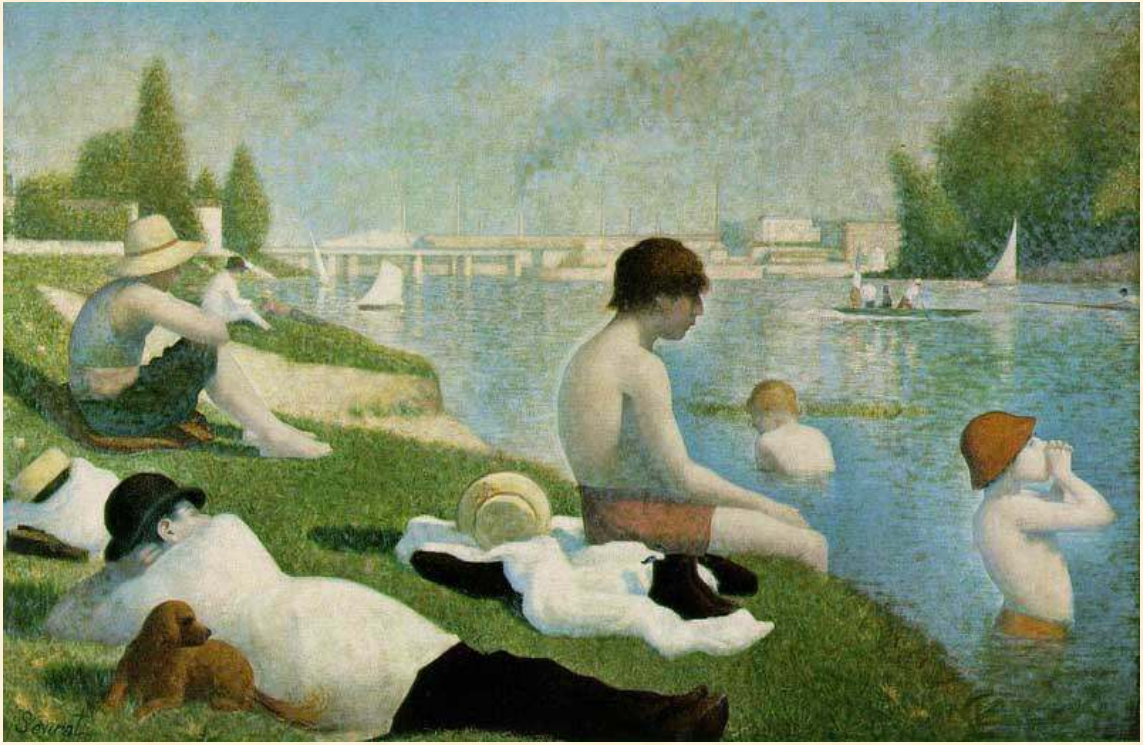
- A perpendicular line from the apex of the triangle to the base would cut the base in golden section.

- Extravagant claims like this one in favor of the golden ratio are hard to uphold. Beware of the phiologists!



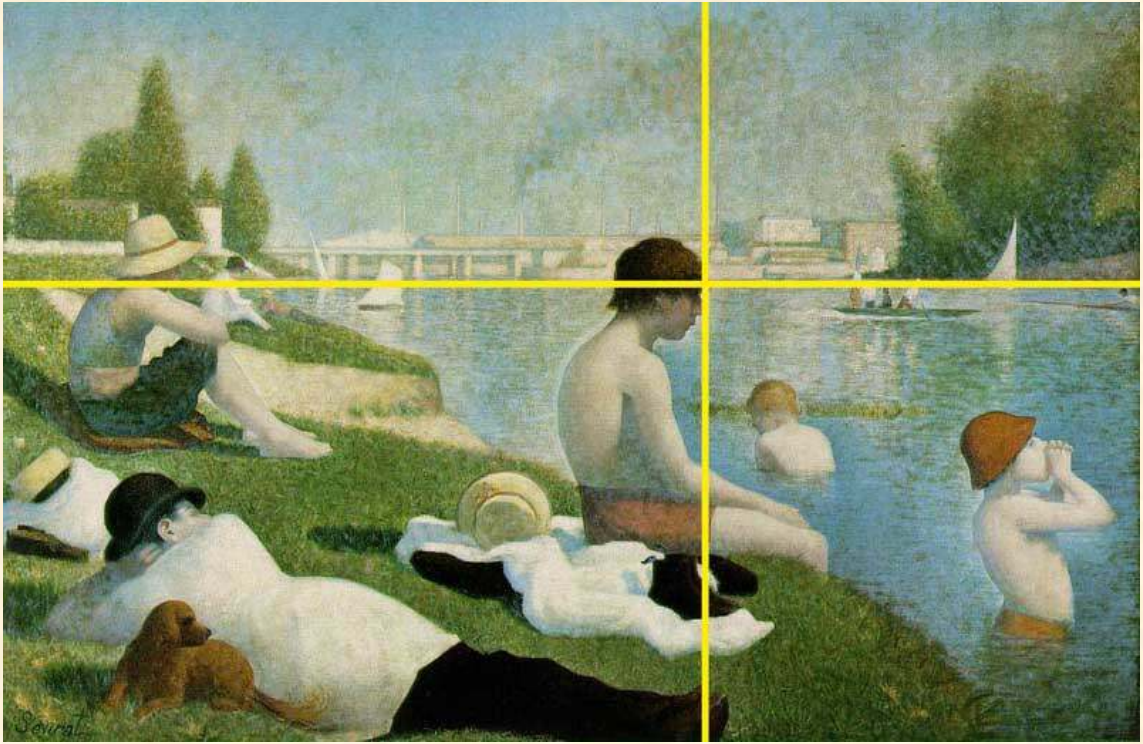
Point master

- According to one art expert, Seurat "attacked every canvas by the golden section". His Bathers ...



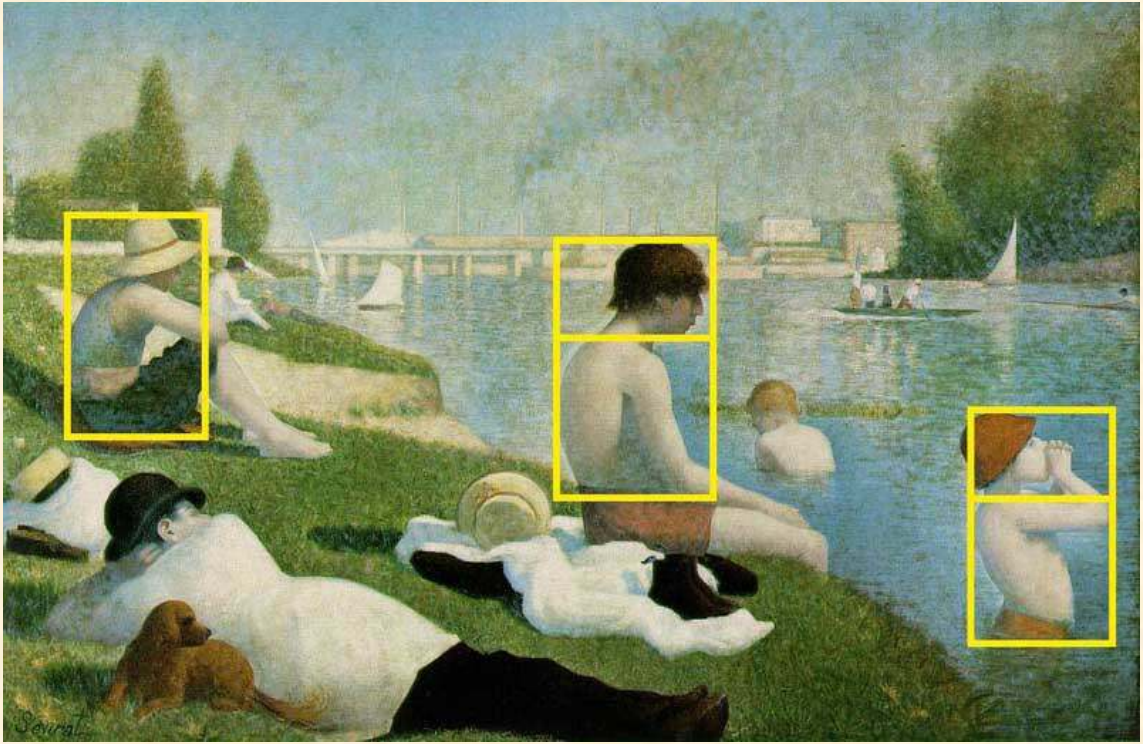
Point master

- … has "obvious" golden subdivisions.



Point master

- Three more golden figures have been added. Can you find more?
- Seurat apparently never mentioned the golden ratio in his writings…

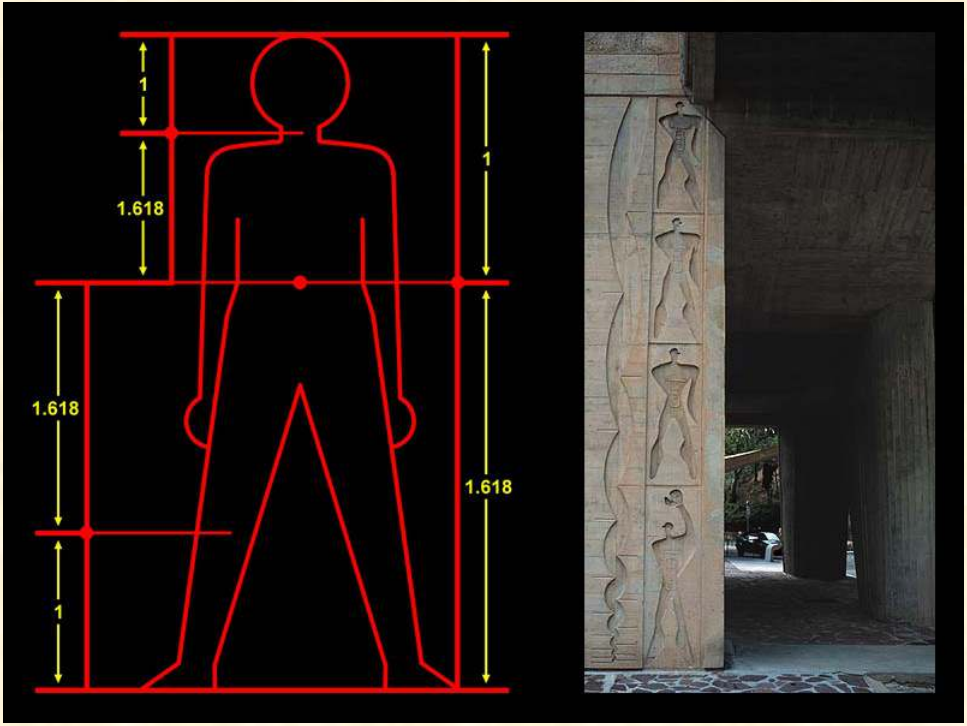


Surreal

- The Sacrament of the Last Supper by Salvador Dali (1904-1989) is framed in a golden rectangle. Golden proportions were also used for positioning the figures. Part of an enormous dodecahedron floats above the table. The polyhedron consists of 12 regular pentagons and has fundamental golden connections.

Golden homes

- The 20th Century architect Le Corbusier (1887-1965) developed a scale of proportions which he called Le Modulor, based on a human body whose height is loosely divided into golden sections.



Golden homes

- The same proportion is to be seen in his modern flats. Le Corbusier felt that human life was "comforted" by mathematics.

Golden braids

- "I have sought to weave an eternal golden braid out of these three strands, Gödel, Escher, Bach, a mathematician, an artist, and a composer."

- This weeks construction follows section 3.1 of the text, ruler and compass constructions.