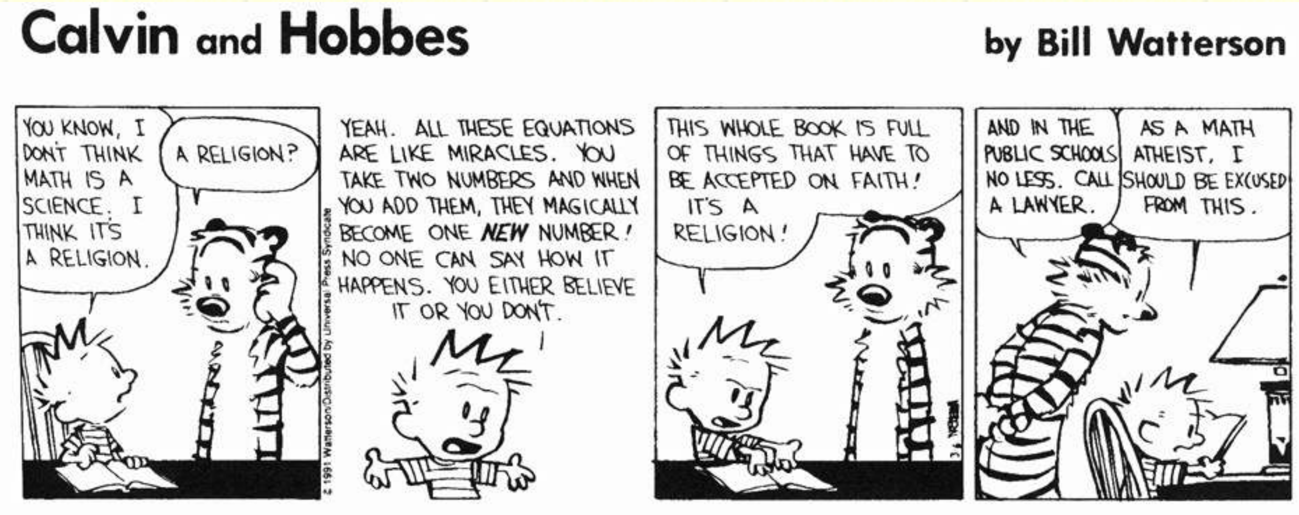

Calvin and Hobbes
by Bill Watterson
YOU KNOW, I DON'T THINK MATH IS A SCIENCE. I THINK IT'S A RELIGION.
A RELIGION?
YEAH. ALL THESE EQUATIONS ARE LIKE MIRACLES. YOU TAKE TWO NUMBERS AND WHEN YOU ADD THEM, THEY MAGICALLY BECOME ONE NEW NUMBER! NO ONE CAN SAY HOW IT HAPPENS. YOU EITHER BELIEVE IT OR YOU DON'T.
© 1991 Watterson/Distributed by Universal Press Syndicate
THIS WHOLE BOOK IS FULL OF THINGS THAT HAVE TO BE ACCEPTED ON FAITH! IT'S A RELIGION!
AND IN THE PUBLIC SCHOOLS NO LESS. CALL A LAWYER.
AS A MATH ATHEIST, I SHOULD BE EXCUSED FROM THIS.

School of the Art Institute of Chicago

Geometry of Art and Nature

Frank Timmes

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flash.uchicago.edu/~fxt/class_pages/class_geom.shtml

Syllabus

1	Sept 03	Basics and Celtic Knots
2	Sept 10	Golden Ratio
3	Sept 17	Fibonacci and Phyllotaxis
4	Sept 24	Regular and Semiregular tilings
5	Oct 01	Irregular tilings
6	Oct 08	Rosette and Frieze groups
7	Oct 15	Wallpaper groups
8	Oct 22	Platonic solids
9	Oct 29	Archimedian solids
10	Nov 05	Non-Euclidean geometries
11	Nov 12	Bubbles
12	Dec 03	Fractals

Sites of the Week

- www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html
- www.branta.connectfree.co.uk/fibonacci.htm

Class #3

- Fibonacci numbers
  - Rabbits, Bees, Sea Shells, Flowers, Seeds, Fruits, Vegetables
  - Relation to the Golden Ratio
  - In art

How many rabbits?

- The original problem that Fibonacci investigated (in the year 1202) was how fast rabbits could breed in ideal circumstances.

- Suppose a newly-born pair of rabbits, one male, one female, are put in a field.
- Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits.

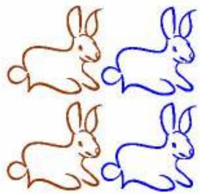
How many rabbits?

- Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on.

- The puzzle that Fibonacci posed: How many pairs will there be in one year?

How many rabbits?

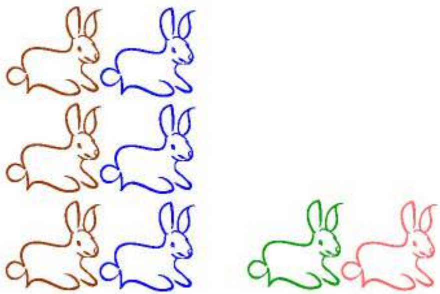
- Start with a pair of baby rabbits.
- At the start of the 2nd month, the first pair mate.



Month	Pairs
1	1
2	1

How many rabbits?

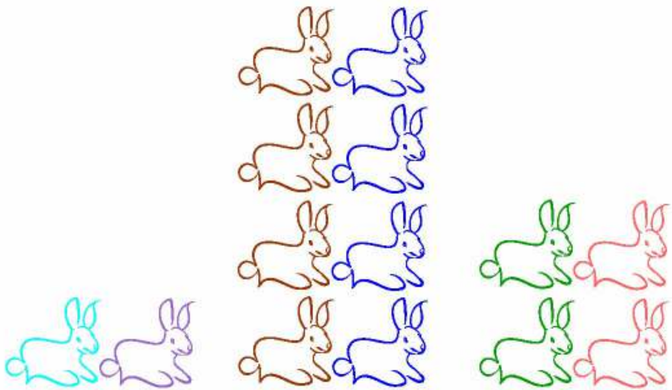
- At the start of the 3rd month the female produces a new pair, so now there are 2 pairs of rabbits in the field.



Month	Pairs
1	1
2	1
3	2

How many rabbits?

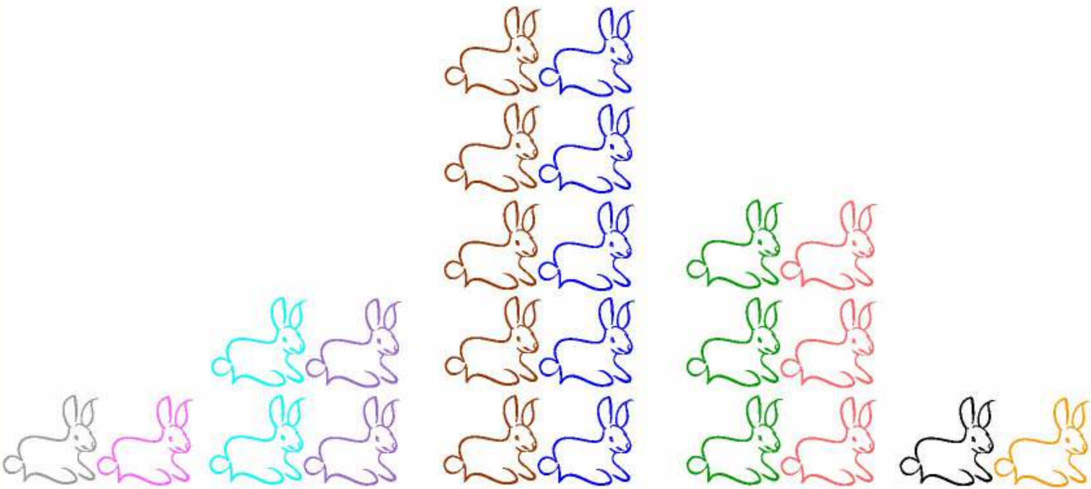
- At the start of the 4th month, the original female produces a second pair, making 3 pairs in all in the field.



Month	Pairs
1	1
2	1
3	2
4	3

How many rabbits?

- At the start of the 5th month, the original female has produced yet another new pair, and the female born two months ago produces her first pair,
  making 5 pairs.



Month	Pairs
1	1
2	1
3	2
4	3
5	5
6	8

How many rabbits?

- The number of pairs of rabbits in the field at the start of each month is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 …

Fibonacci numbers

- The numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 …

- Can you see how the series is formed and how it continues?

- The Fibonacci series is formed by starting with 0 and 1 and then adding the latest two numbers to get the next one.

$$\mathrm{Fib}(0) = 0$$

$$\mathrm{Fib}(1) = 1$$

$$\mathrm{Fib}(n) = \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) \qquad n > 2$$

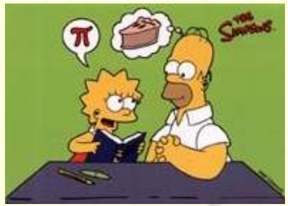
Fibonacci numbers

- The rabbits problem isn't very realistic.
    - It implies that brother and sisters mate, which, genetically, leads to problems.
    - Another problem which isn't true to life, is that each birth is of exactly two rabbits, one male and one female.

Chicago artist Eduardo Kac wanted to use a rabbit named Alba in an exhibition. The French scientists who created her blocked the plan. Alba had been modified with the GFP jellyfish gene.

Fibonacci numbers

- But Fibonacci did what mathematicians often do at first, simplify the problem and see what happens.



- The series bearing his name has lots of other interesting and practical applications, as we'll see.

- Let's start by looking at a real-life situation that is exactly modeled by Fibonacci's series; honeybees.

Honeybee geneology

- There are over 30,000 species of bees and in most of them the bees live solitary lives.

- The one most of us know best is the honeybee and it, unusually, lives in a colony called a hive and they have an unusual family tree.

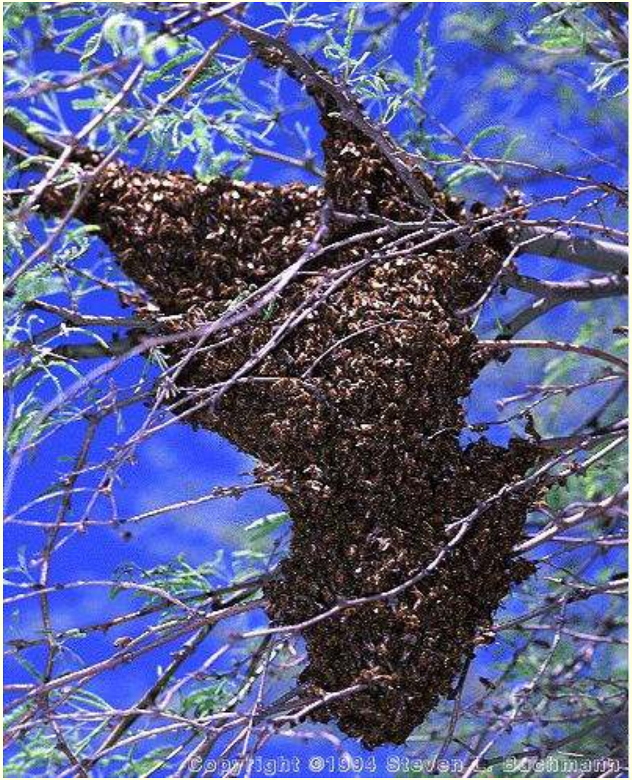
Honeybee geneology

- Some unusual facts about honeybees:
- Not all of them have two parents!
- In a colony of honeybees there is one special female called the queen.

Honeybee geneology

- Worker bees are female too but unlike the queen bee, they produce no eggs.

- Females are produced from fertilized eggs when the queen has mated with a male. So, females have two parents.



Honeybee geneology

- Females usually end up as worker bees, but some are fed with a special substance called royal jelly which makes them grow into queens.

- The new queen starts a new colony when some of the bees swarm and leave their hive in search of a place to build a new nest.

Honeybee geneology

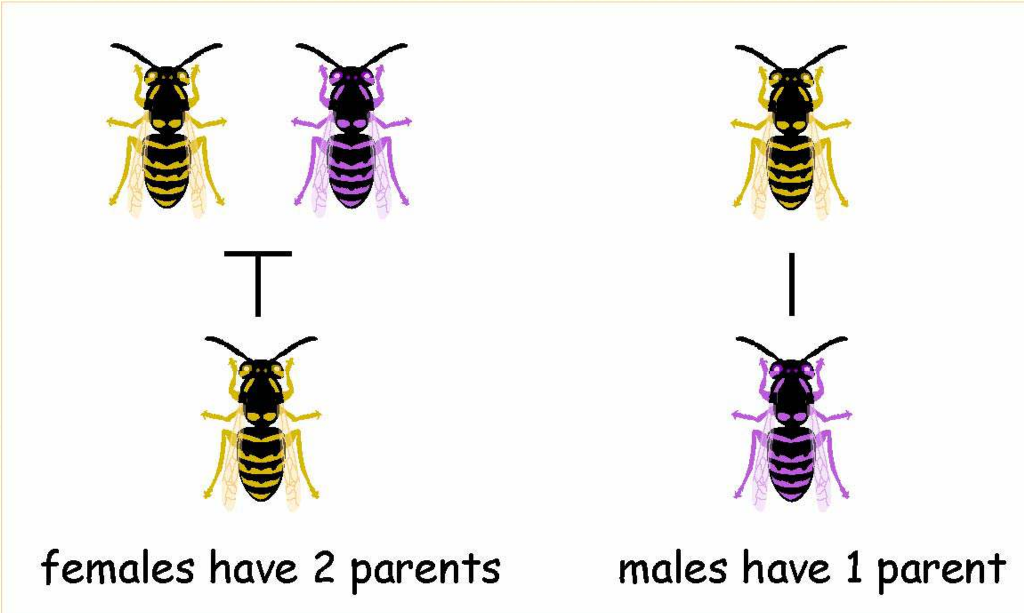
- Drone bees are male and do no work.

- Males are produced by the queen's unfertilized eggs, so male bees only have a mother but no father!

The Drone Bee - His only reason for living is to mate with a virgin queen bee and then he dies. He does not even have a stinger. If he happens to live to the Fall, the worker bees will force him out of the hive and since he can not make food for himself he will die.

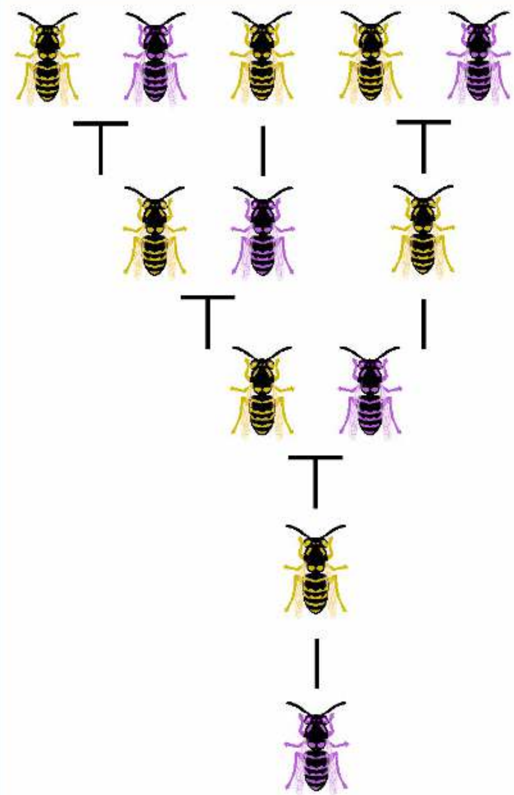
Honeybee geneology

- So, female bees have two parents, a male and a female whereas male bees have just one parent, a female.



Honeybee geneology

- Let's look at the family tree of a drone:
  - He has 1 parent, a female.
  - He has 2 grand-parents, since his mother had two parents.
  - He has 3 great-grand-parents: his grand-mother had two parents but his grand-father had only one.
  - How many great-great-grand parents does he have?



Drone Bee Geneology

Honeybee geneology

- Again we see the Fibonacci numbers!

	Parents	Grand parents	Great grand parents	2• Great grand parents	3•Great grand parents
Male	1	2	3	5	8
Female	2	3	5	8	13

Fibonacci and the golden ratio

- If we divide each Fibonacci number, (1, 1, 2, 3, 5, 8, 13, 21, 34, 55 ...) by the number before it, we will find the following series of numbers:

1/1 = 1

2/1 = 2

3/2 = 1.5

5/3 = 1.66666

8/5 = 1.6

13/8 = 1.625

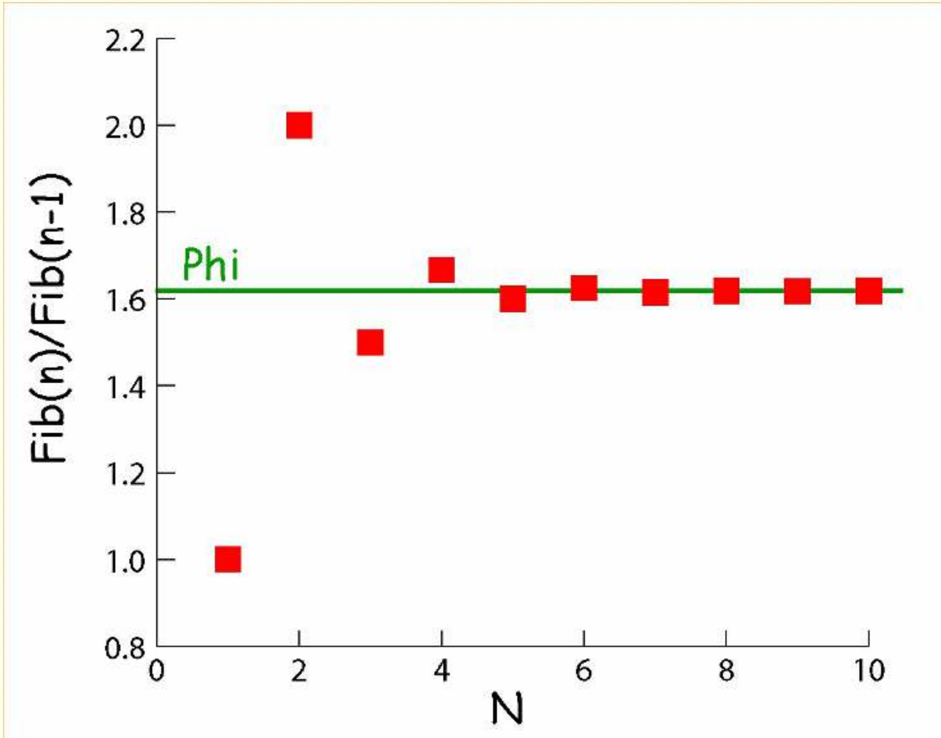
21/13 = 1.61538

34/21 = 1.61904

55/34 = 1.61764

If we plot the ratios on a graph:

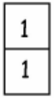
Fibonacci and the golden ratio



- The ratio settles down to the golden section, Phi = 1.61804.

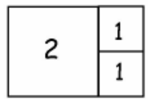
Fibonacci rectangles

- We can make another picture showing the Fibonacci numbers 1,1,2,3,5,8,13,21,.. if we start with two small squares of size 1 next to each other.



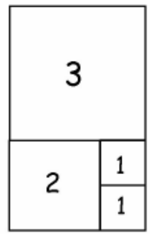
Fibonacci rectangles

- On top of both of these draw a square of size 2 (= 1 + 1).



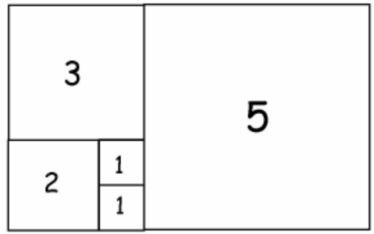
Fibonacci rectangles

- We now draw a new square having sides 2 + 1 = 3 units long.



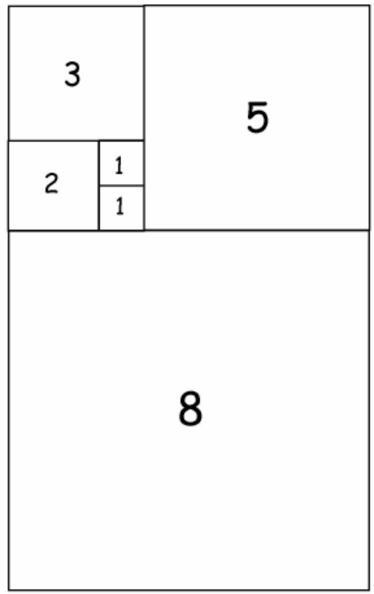
Fibonacci rectangles

- Then another with sides 2 + 3 = 5 units long.



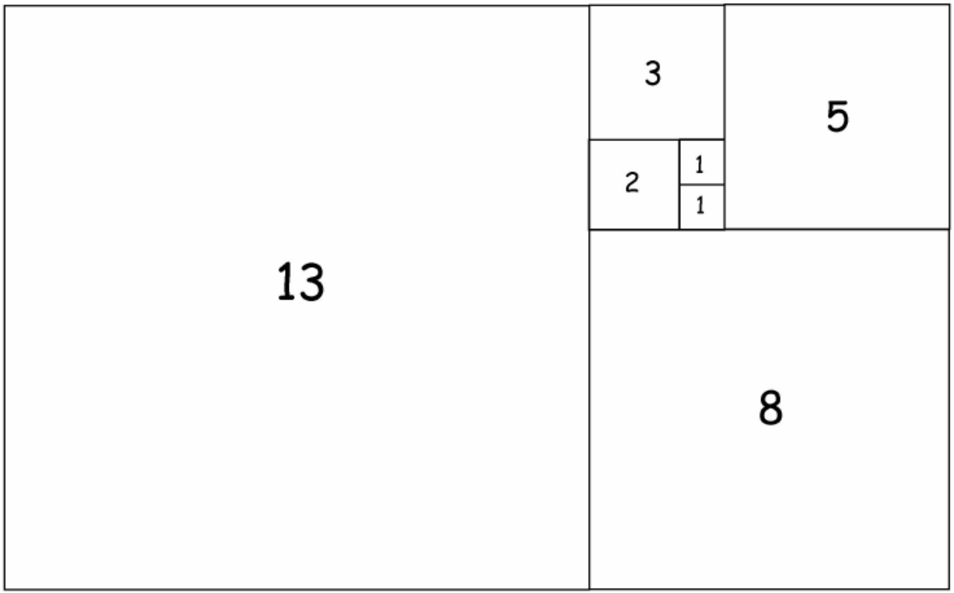
Fibonacci rectangles

- We continue adding squares around the picture, each new square having a side which is as long as the sum of the latest two square's sides.



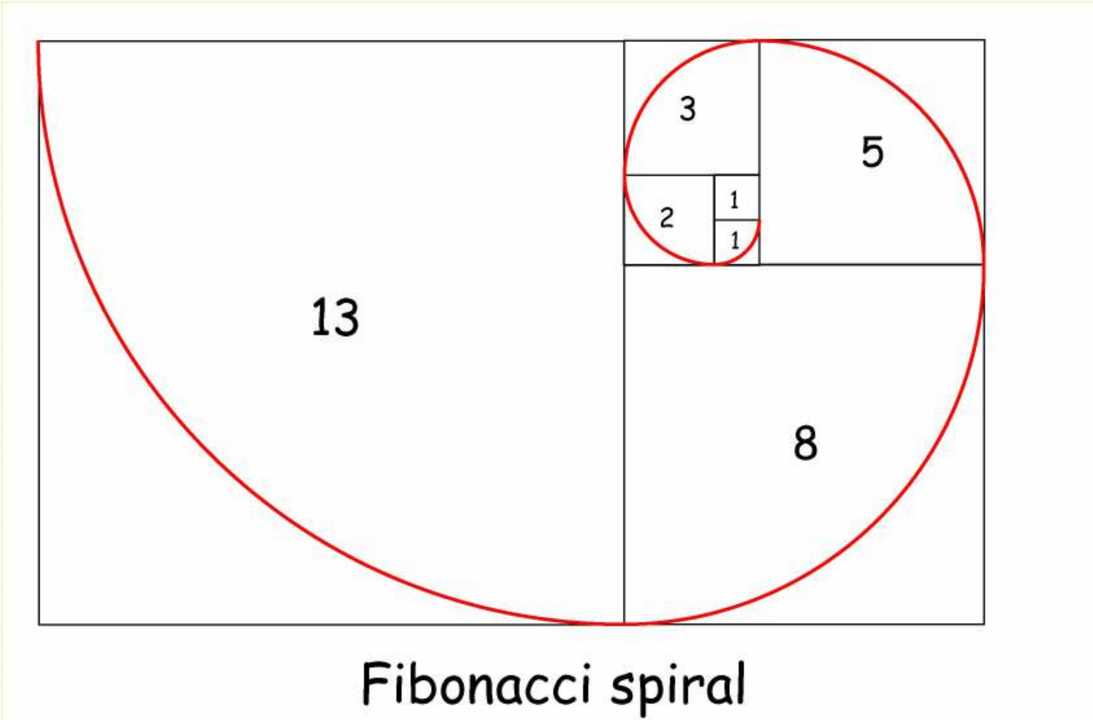
Fibonacci rectangles

- This set of rectangles whose sides are two successive Fibonacci numbers in length and are composed of squares with sides which are Fibonacci numbers, are called Fibonacci Rectangles.



Fibonacci spirals

- By putting quarter circles in each square, we draw the Fibonacci Spiral.



Fibonacci spirals

- Similar curves occur in nature as the shape of a snail shell or some sea shells.

Fibonacci spirals

- Whereas the Fibonacci spiral increases in size by a factor of Phi in a quarter of a turn, the Nautilus spiral takes a whole turn before points move a factor of 1.618 from the center.

Phyllotaxis

- On many plants, the number of petals is a Fibonacci number.
- One petal …

white calla lily

Phyllotaxis

- … and two petals are not that common.

euphorbia

Phyllotaxis

- Three petals are more common (lily, iris).

trillium

Phyllotaxis

- There are hundreds of species, both wild and cultivated, with five petals (buttercup, wild rose, larkspur).

columbine

Phyllotaxis

- Eight-petalled flowers are not so common as five,
  but there are quite a number of well-known species with eight (delphiniums).

bloodroot

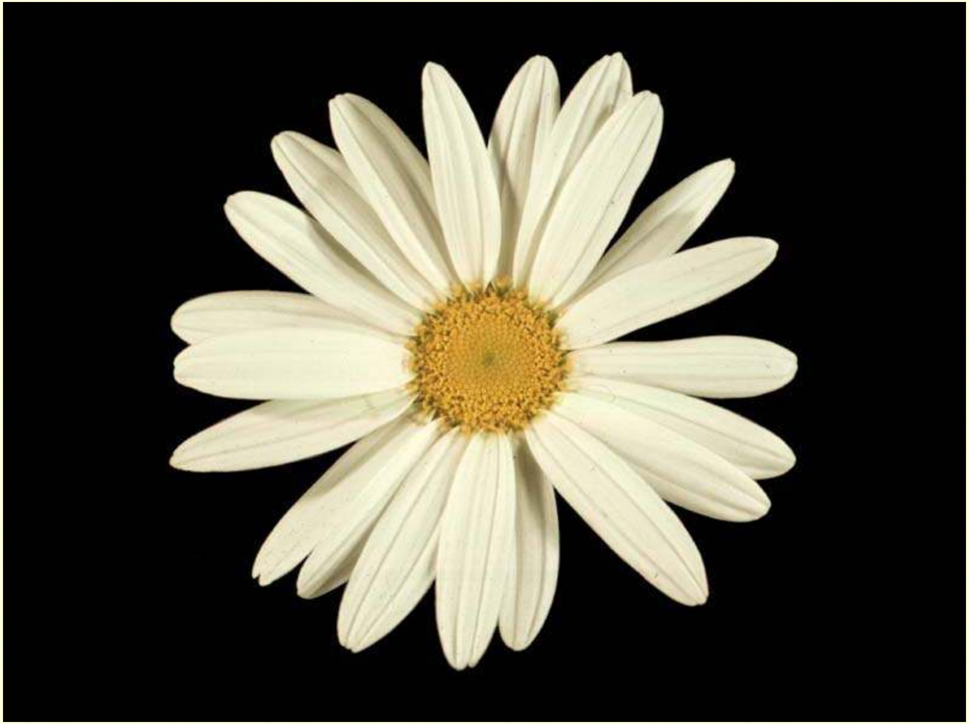
Phyllotaxis

- Thirteen (ragwort, corn marigold, cineraria)...

Black-eyed susan

Phyllotaxis

- … twenty-one (astor, chicory) and thirty-four petals (plantain, pyrethrum) are also quite common. Daisies illustrate the Fibonacci sequence extremely well.



Shasta daisy

Phyllotaxis

- Ordinary field daisies have 34 petals, something to consider when playing "loves me, loves me not". Some daisies may have 33 petals, some 35, but the average is 34 petals.

field daisies

Phyllotaxis

- Daisies with 55 or 89 petals are also common (michaelmas, asteraceae family).

michaelmas

Seed heads

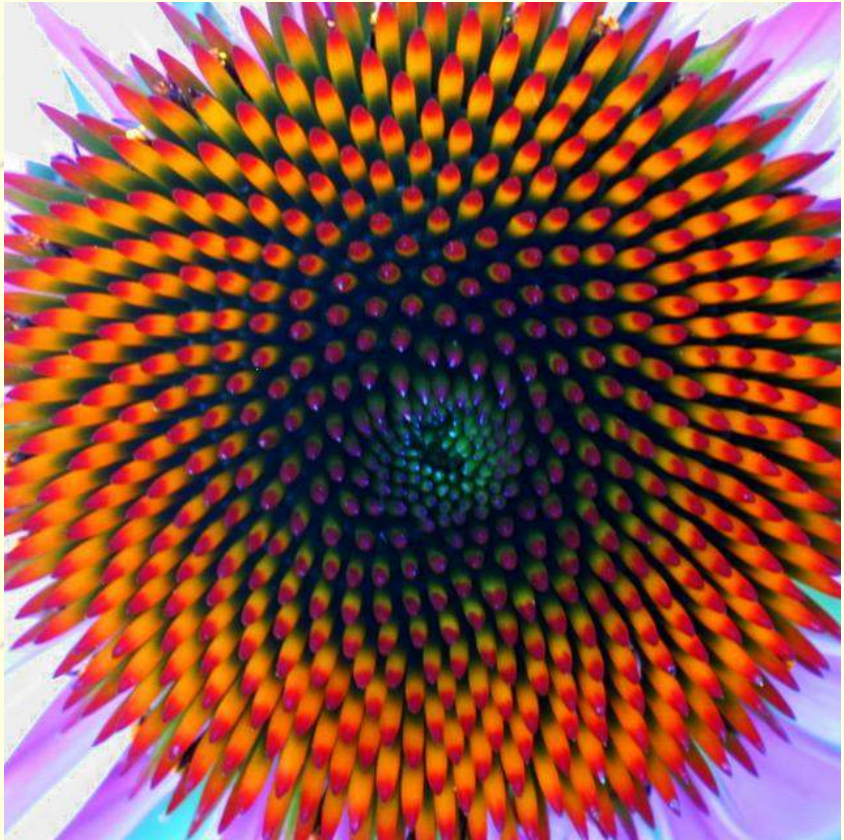
- Fibonacci numbers can also be seen in the arrangement of seeds on flower heads.

- The daisy seedhead below is about 2 cm across and native to the Illinois prairie.

- The orange "petals" seem to form spirals curving both left and right.

Seed heads

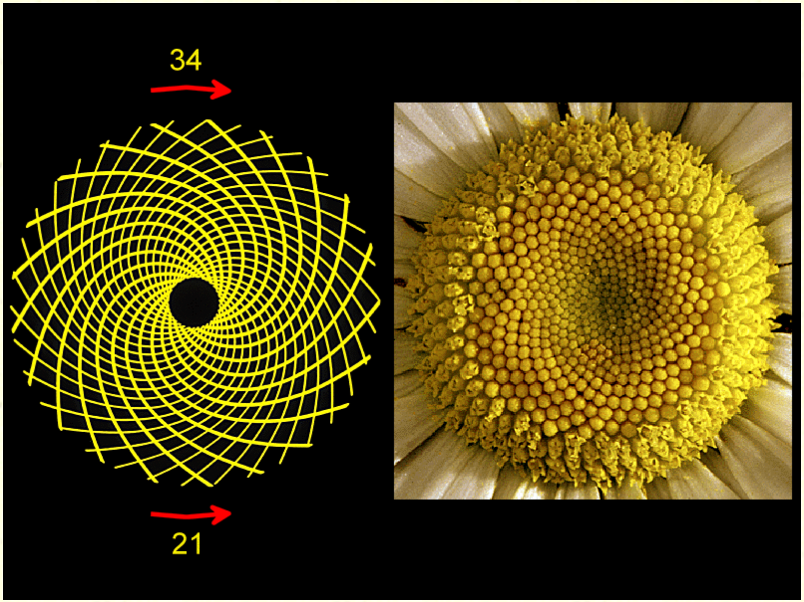
- Near the edges you can count 55 spirals going right.
- A little further towards the center and you can count 34 spirals going left.
- This pair of numbers are Fibonacci neighbors.



Seed heads

- "Curvier" spirals appear near the center, flatter spirals appearing the farther away from the center we go.

- The number of spirals we see, in either direction, is larger for larger seed heads than for small.



Cone head

- Let's also look at pinecones.

Cone head

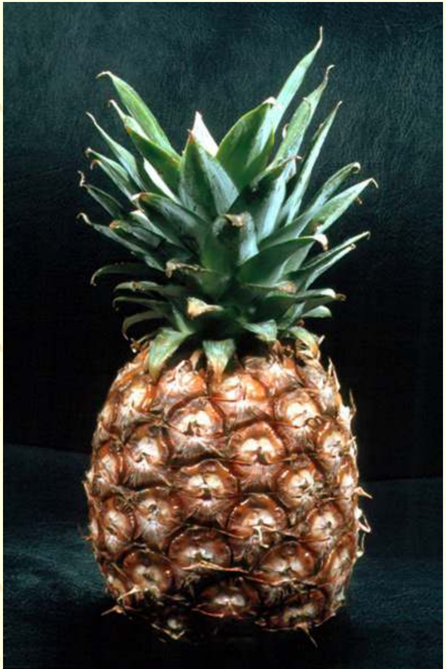
- The seed-bearing scales of a pinecone are really modified leaves, crowded together and in contact with a short stem.
- We can detect two prominent arrangements of ascending spirals growing outward from the point where it is attached to the branch.

- Eight spirals can be seen going up in a clockwise direction …

- … while thirteen spirals ascend more steeply in an anticlockwise direction.

Pineapples

- Pineapple scales are also patterned into spirals and because the scales are roughly hexagonal in shape, three distinct sets of spirals may be observed.



One set of 5 spirals ascends at a shallow angle to the right,

a second set of 8 spirals ascends more steeply to the left,

and a third set of 13 spirals ascends very steeply to the right.

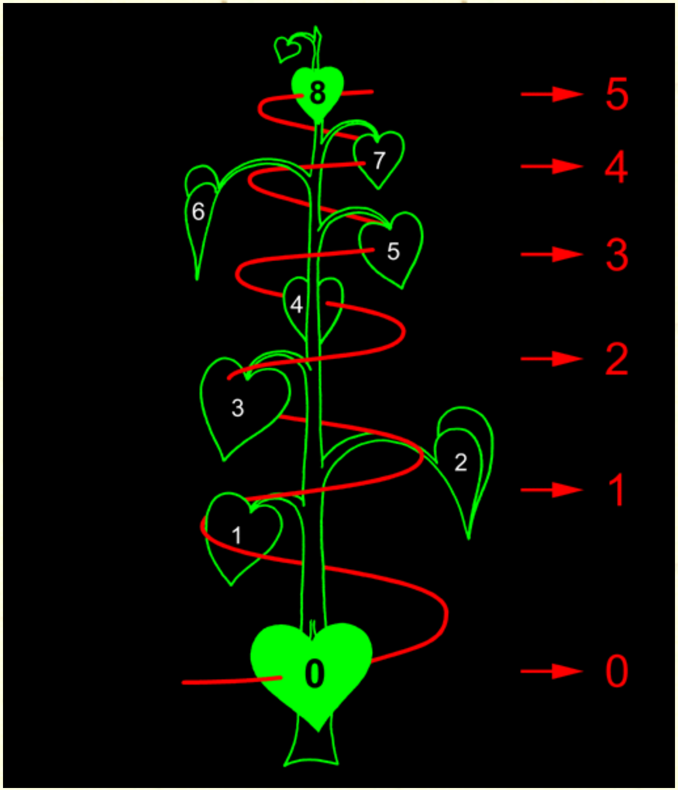
Cauliflowers

- Note how a cauliflower is roughly a pentagon in outline.
- Looking carefully, you can see a center point, where the florets are smallest. Note the florets are organized in spirals around this center in both directions.

Leaf arrangement

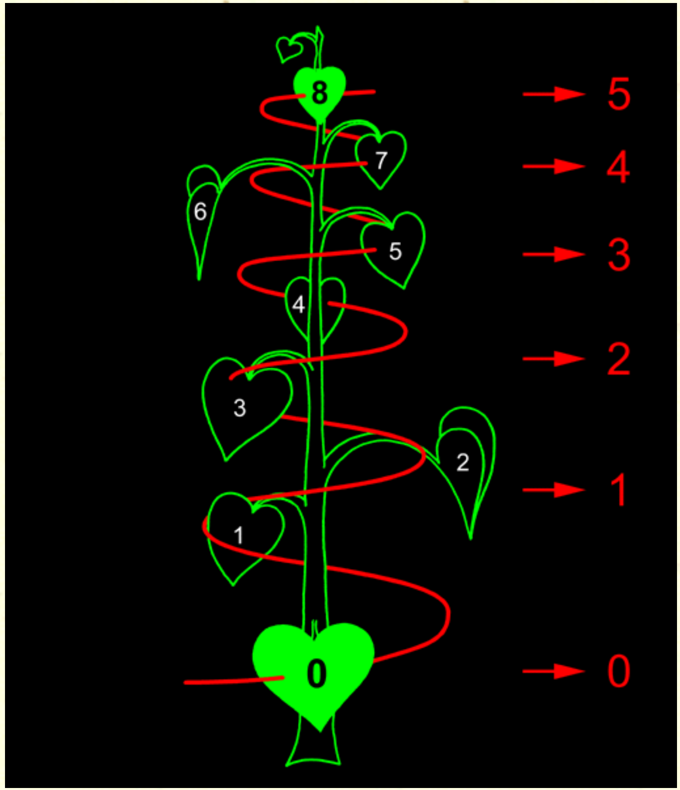
- Many plants show the Fibonacci numbers in the arrangements of the leaves around their stems.

- Counting the number of times we go around the stem in one direction until we encounter a leaf directly above the starting one gives one Fibonacci number.



Leaf arrangement

- Count in the other direction, and we get a different Fibonacci number of turns for the same number of leaves.
- The number of turns in each direction and the number of leaves are three consecutive Fibonacci numbers!

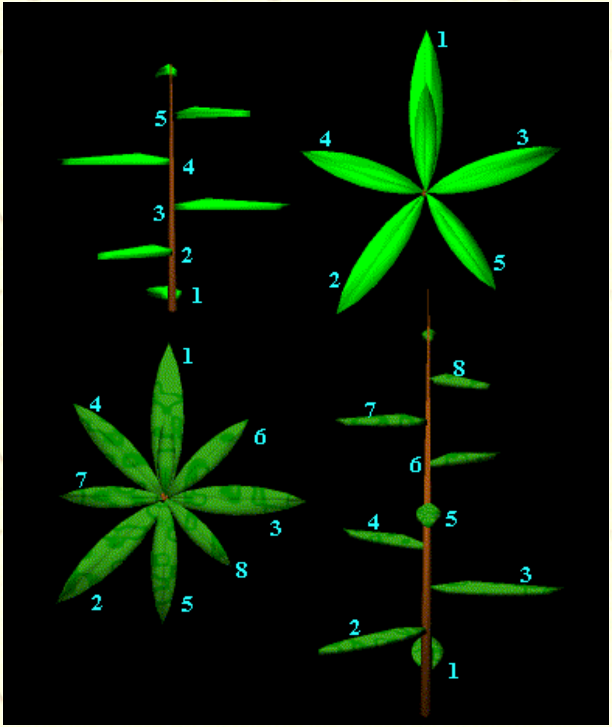


Leaf arrangement

- For the top plant, we make 3 clockwise rotations before meeting a leaf directly above the first, passing 5 leaves on the way.

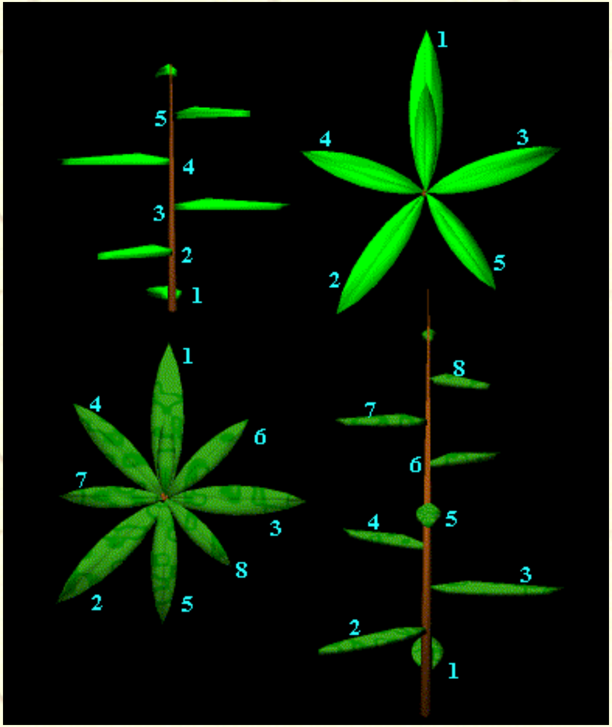
- If we go counter-clockwise, we need only 2 turns.

- 2, 3 and 5 are consecutive Fibonacci numbers.



Leaf arrangement

- For the lower plant, we have 5 clockwise rotations passing 8 leaves, and only 3 rotations in the anti-clockwise direction.



- This time 3, 5 and 8 are the consecutive numbers in the Fibonacci sequence.

Leaf arrangement

- About 90% of all plants exhibit leaf patterns involving the Fibonacci numbers.

- Some common trees with their Fibonacci leaf arrangement numbers are:

1/2 elm, linden, lime, grasses

1/3 beech, hazel, grasses, blackberry

2/5 oak, cherry, apple, holly, plum, common groundsel

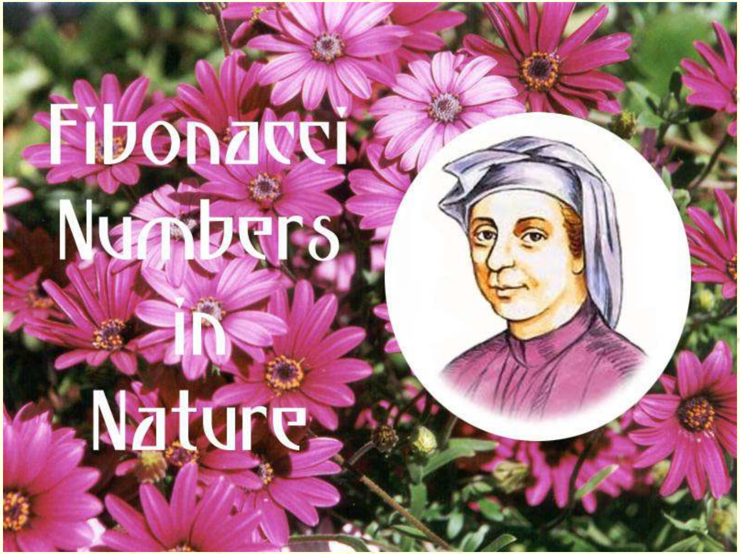
3/8 poplar, rose, pear, willow

5/13 pussy willow, almond

where t/n means there is t turns for n leaves;
each leaf makes t/n of a turn after the last leaf.

Why?

- The Fibonacci number patterns occur so frequently in nature that one may ask if it is a "law of nature".
- Four-petalled flowers are not so rare as the four-leaf clover is reputed to be, and deviations, sometimes large ones, from Fibonacci patterns have been found.
- The phenomenon isn't a physical law. It is, however, a prevalent tendency.



Why?

- We saw the Fibonacci numbers appear in (idealized) rabbit and bee populations, petals of a flower, leaves around branches, seeds on seed-heads, and in fruit and vegetables like pineapples and cauliflowers.

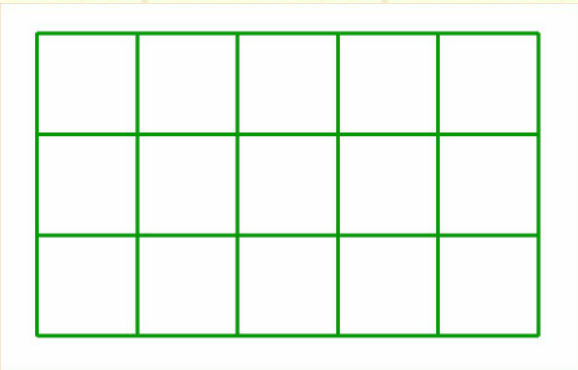
- We explained why they appear in the rabbit and bee populations, but why does nature use the Golden ratio and Fibonacci numbers in so many plants?

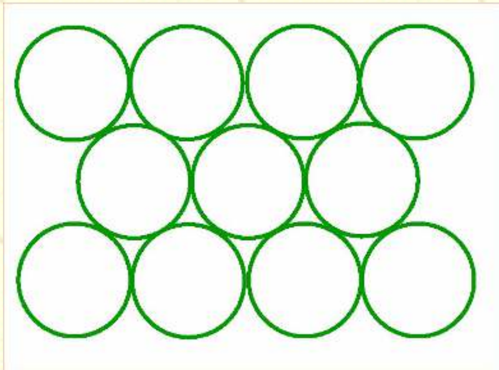
- The answer lies in packings - the best arrangement of objects to minimize wasted space; or to say it another way, maximize useful space.

Packings

- If you were asked what was the best way to pack objects your answer would depend on the shape of the objects since....

- ...square objects would pack most closely in a rectangular array,





- ... whereas round objects pack better in a hexagonal arrangement.

Packings

- Seeds are mostly round, so why doesn't nature use hexagonal arrangements for seedheads?

- Although hexagonal symmetry is the best packing for circular seeds, it doesn't answer how leaves should be arranged around a stem or how to pack flower-heads with seeds that grow in size.

Packings

- Nature seems to use the same pattern to place seeds on a seedhead, arrange petals around the edge of a flower, and place leaves round a stem.

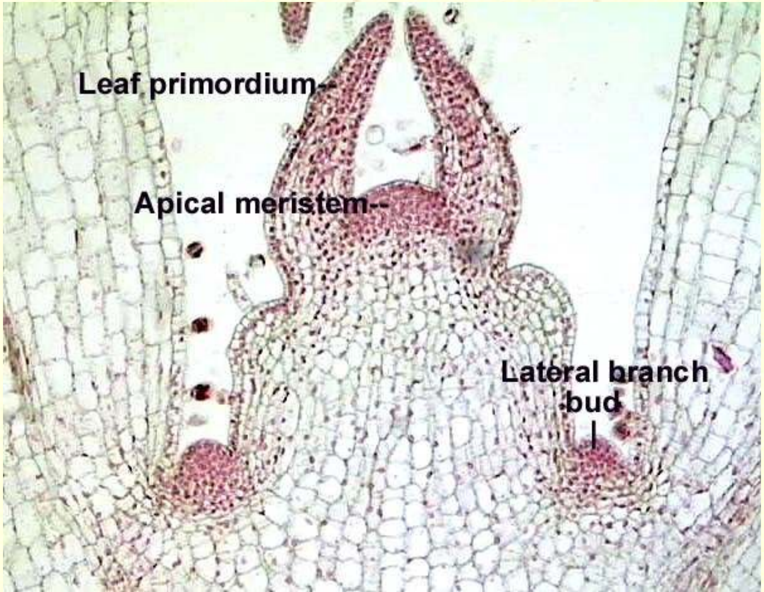
- Not only that but all these patterns maintain their optimal packing structure as the plant continues to grow.

- This is a lot to ask of a single process!
  How do plants grow and maintain this optimal design??

The meristem

- Botanists have shown that plants grow from a single tiny group of cells right at the tip of any growing plant, called the meristem.

- A meristem is at the end of each branch or twig where new cells are formed.



- Once formed at the meristem, the new cells grow in size.
  So, the growing point keeps moving upward and outward.

The meristem

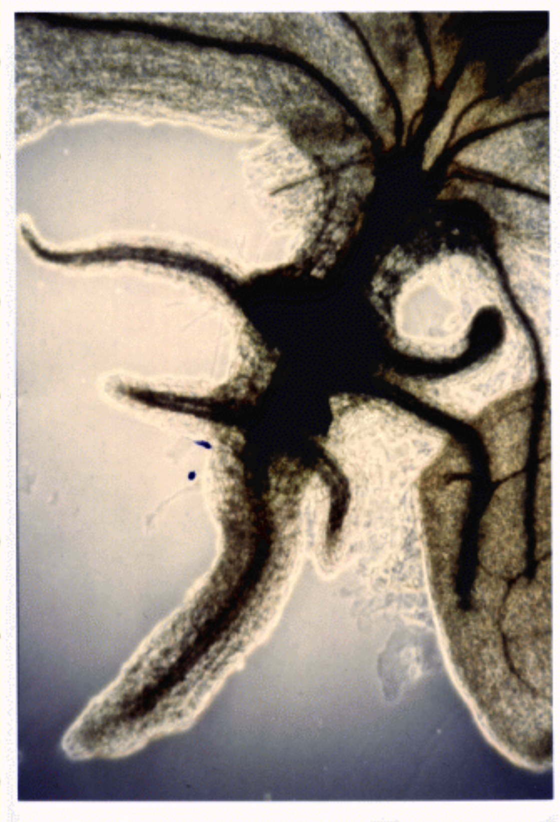
- A meristem moves in a circular pattern in space, or in a helical fashion in spacetime.
- The stem turns by an angle and then a new cell appears, turns again and another new cell is formed, and so on.
- These cells may then become a new branch, petals, seeds, or whatever.

The meristem

- The amazing thing is that a single fixed angle can produce this optimal design no matter how big the plant grows.

- Once an angle is fixed for a leaf, that leaf will least obscure the leaves below and be least obscured by any future leaves above it.

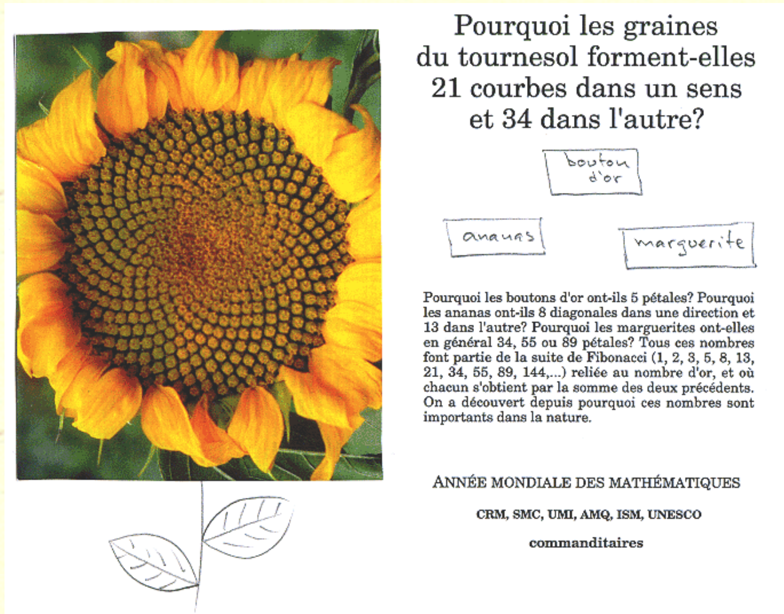
- Once a seed is positioned on a seedhead, it is pushed out in a straight line by other new seeds, retaining its original angle on the seedhead.



One fixed angle

- All this can be done with a single fixed angle of rotation between new cells? Yes! This was suspected by people as early as the last century.

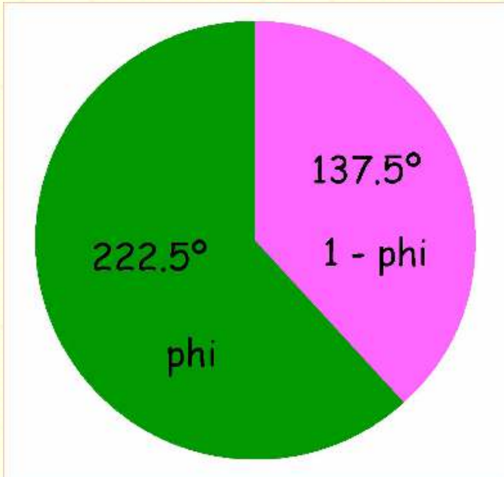
- The principle that a single angle produces uniform packings no matter how much growth occurs was only proved mathematically in 1993 by Douady and Couder, two French mathematicians.



One fixed angle

- You've probably guessed what the fixed angle of turn is: phi turns per new cell, or Phi cells per turn.

- A new leaf, seed, or petal is made every 0.618 turns. In terms of degrees, this is 0.618 • 360° = 222.5°. However, we tend to "see" the smaller angle of 137.5°.



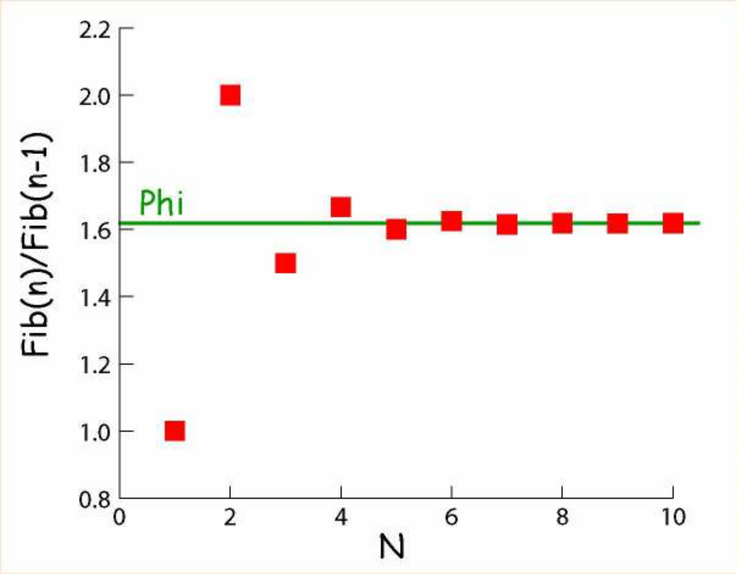
One fixed angle

- If there phi = 0.618 turns per leaf, then we get a packing such that each leaf gets the maximum exposure to light, casting the least shadow on the others.
- This also gives the largest possible area exposed to falling rain so the rain is directed back along the leaf and down the stem to the roots.
- For flowers, it also gives the best exposure to attract insects for pollination.
- The whole of the plant seems to produce its leaves, flower head petals, and then seeds based upon the golden number.

One fixed angle

- But why do the Fibonacci numbers appear as leaf arrangements, and as the number of spirals on seedheads?

- Because the Fibonacci numbers form the best integer fractions to the golden ratio!



- Let's see why phi is the best angle to use.

Exact is fruitless

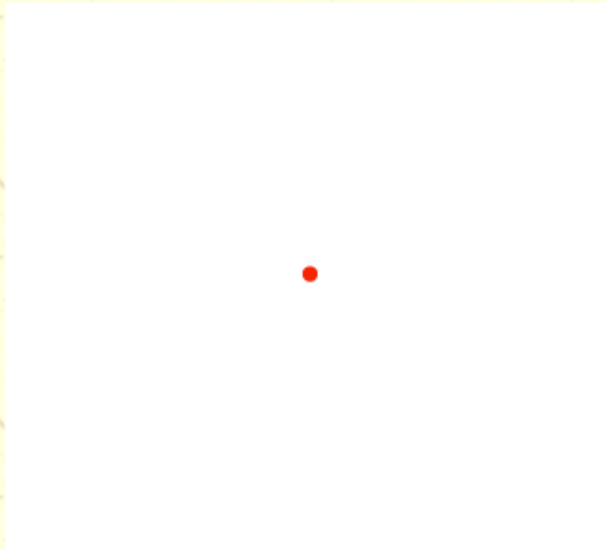
- Why not 0.5 or 0.48 or 1.6 or some other number of turns per new cell?

- First, agree that turning 0.6 of a turn is exactly the same as turning 1.6 turns or 2.6 turns or 12.6 turns because the position of the point remains the same.
  We can ignore the whole number part and only examine the fractional part.

- In terms of seeds - which develop into fruit - what are fruitful numbers?
  It turns out that numbers which are simple fractions are not good choices.

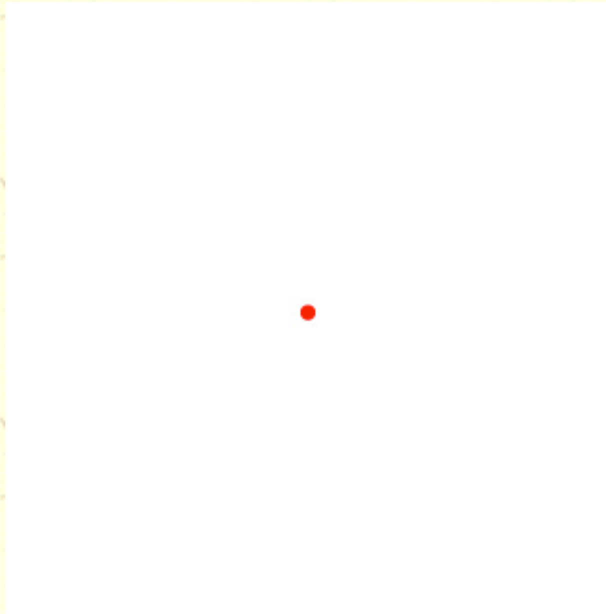
Exact is fruitless

- With a simple number like 0.5 turns per seed we get just 2 "arms" and the seeds use the space very inefficiently: the seedhead is long and floppy.
- A circular seedhead is more compact and would have better mechanical strength and so be better able to withstand wind and heavy rain.
- You can see the new seeds appearing from the central growing point as the older ones are pushed outwards in a straight line.



Exact is fruitless

- Here is 0.48 turns between seeds.
- The seeds seem to be sprayed from two revolving "arms". Since 0.48 is a bit less than 0.5, the "arms" seem to rotate backwards a bit each time.
- If we used 0.52 seeds per turn, we would be a little in advance of half a turn and the final pattern would be a mirror-image.

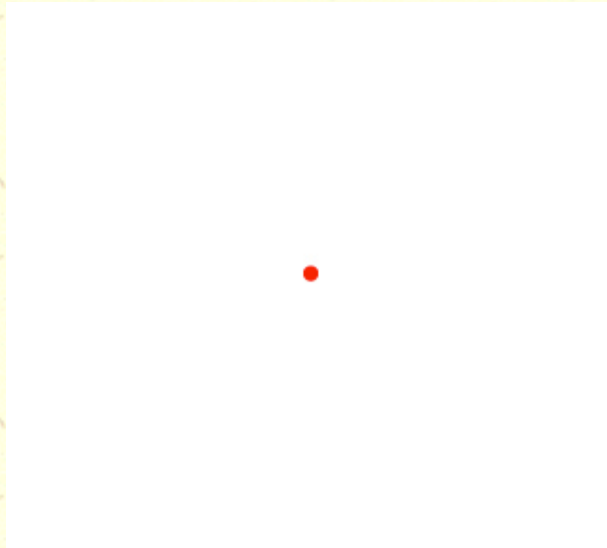


Exact is fruitless

- What do you think will happen with 0.6 turns between seeds?

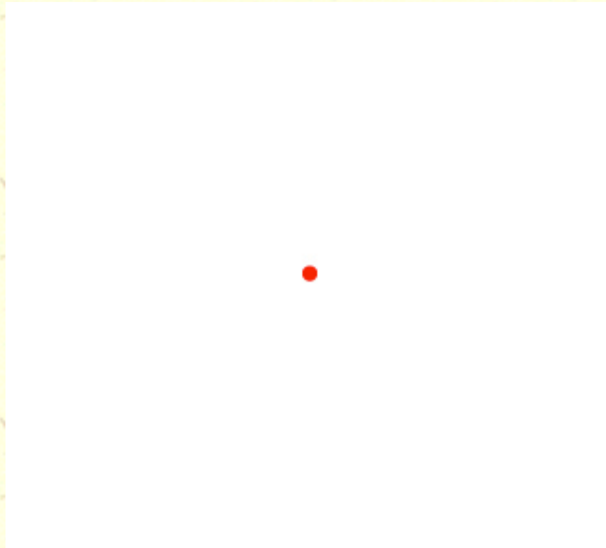
- Did you expect it to be so different? Notice how the seeds are not equally spaced, but fairly soon settle down to 5 "arms".

- This happens because 0.6 = 3/5.
  So every 3 turns will have produced exactly 5 seeds, and the sixth seed will be at the same angle as the first and so on.



Exact is fruitless

- Using a value closer to phi, namely 0.61, is better, but that there are still large gaps between the seeds nearest the center, so the space is not best used.



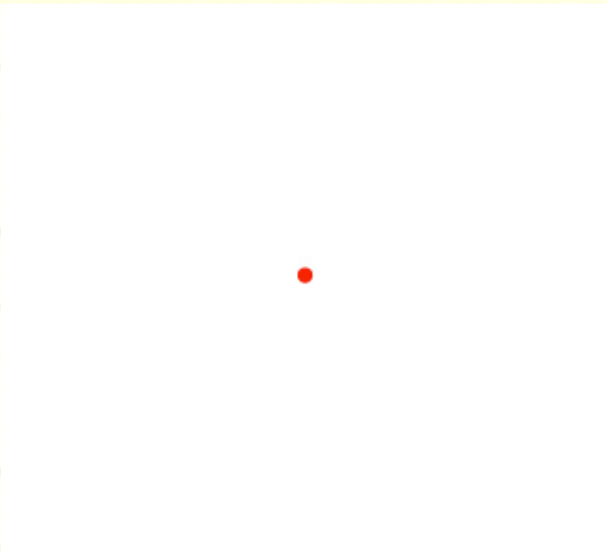
- In fact, any number that can be written as an exact ratio (a rational number) isn't good as a turn-per-seed angle.

- If we use t/n as our angle, then we will end up with n straight arms, the seeds being placed every t^{th} arm.

Irrationals are best

- So what is a "good" value? One that is NOT an exact ratio since very large seed heads will eventually end up with seeds in straight lines.

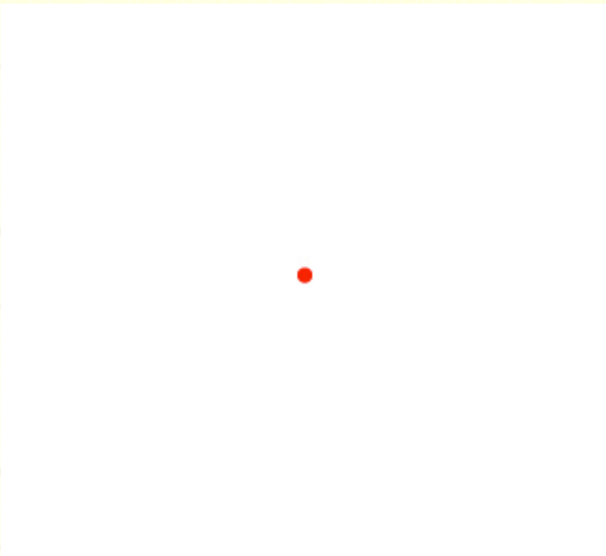
- Numbers which cannot be expressed exactly as a ratio are called irrational numbers and this description applies to such values as pi (3.14159) …



The pi animation has 7 arms since its 0.14159 turns per seed is a bit less than 5/7. If we took more and more seeds, the spirals alter and we would get better approximations to pi.

Irrationals are best

- … and e (2.71827).



The e animation also has 7 arms since its 0.71827 turns per seed is a bit more than 5/7.

So the "arms" bend in the opposite direction to that of pi's.

Again, more seeds would destroy the pattern as we got closer to e.

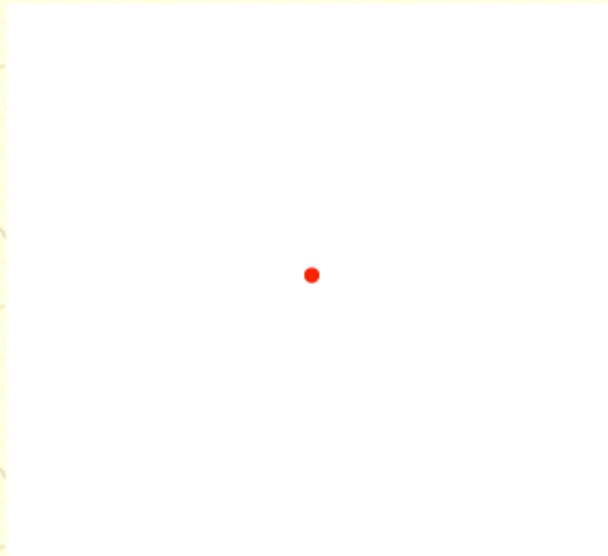
Most fruitful

- What is "the best" irrational number?
  One that never settles down to a rational approximation for very long.

- The simplest such numbers are Phi and phi, whose rational approximations are:
  phi: 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, 13/21, ...
  Phi: 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, ...

- Which is why you see Fibonacci sequences in meristems!

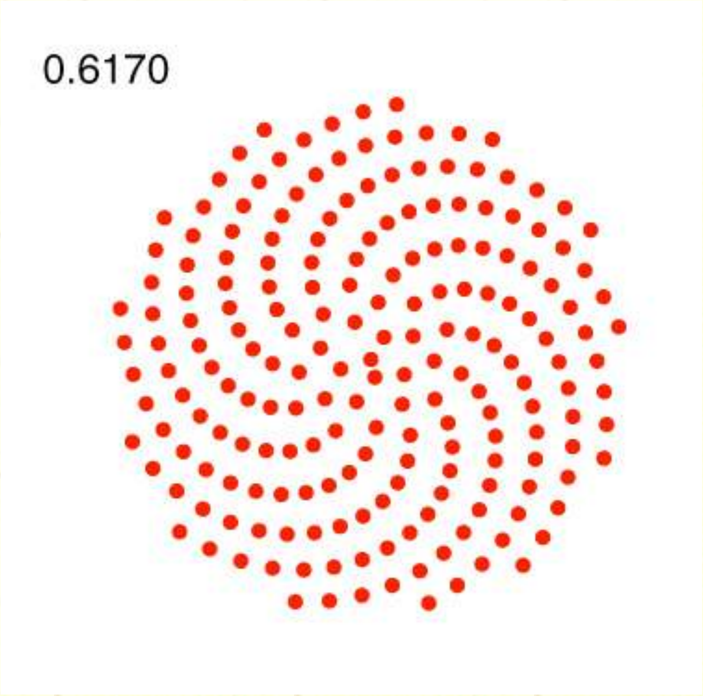
- No matter how big the seed head gets, the seeds are always equally spaced.



Most fruitful

- This movie which shows various turns per seed values near phi.

- This shows that there are always gaps towards the outer edge of the seedhead and that phi gives the best value for all sizes of the seedhead.



Son of Bonacci

- The "greatest European mathematician of the middle ages". His full name was Leonardo of Pisano, since he was born about 1175 in Pisa.

- Pisa was an important commercial town in its day, and Leonardo's father, Guglielmo Bonaccio, was a customs officer in the North African town of Bougie.

Son of Bonacci

- Leonardo grew up with a North African education under the Moors and later traveled extensively around the Mediterranean coast.

- He met many merchants and learned of their systems for doing arithmetic, and soon realized the many advantages of the "Hindu-Arabic" system over the “Roman” system, and introduced it into Europe.

Son of Bonacci

- He called himself Fibonacci, short for “filius Bonacci” meaning “son of Bonacci”. Sometimes he used Leonardo Bigollo since, in Tuscany, bigollo means a traveler.

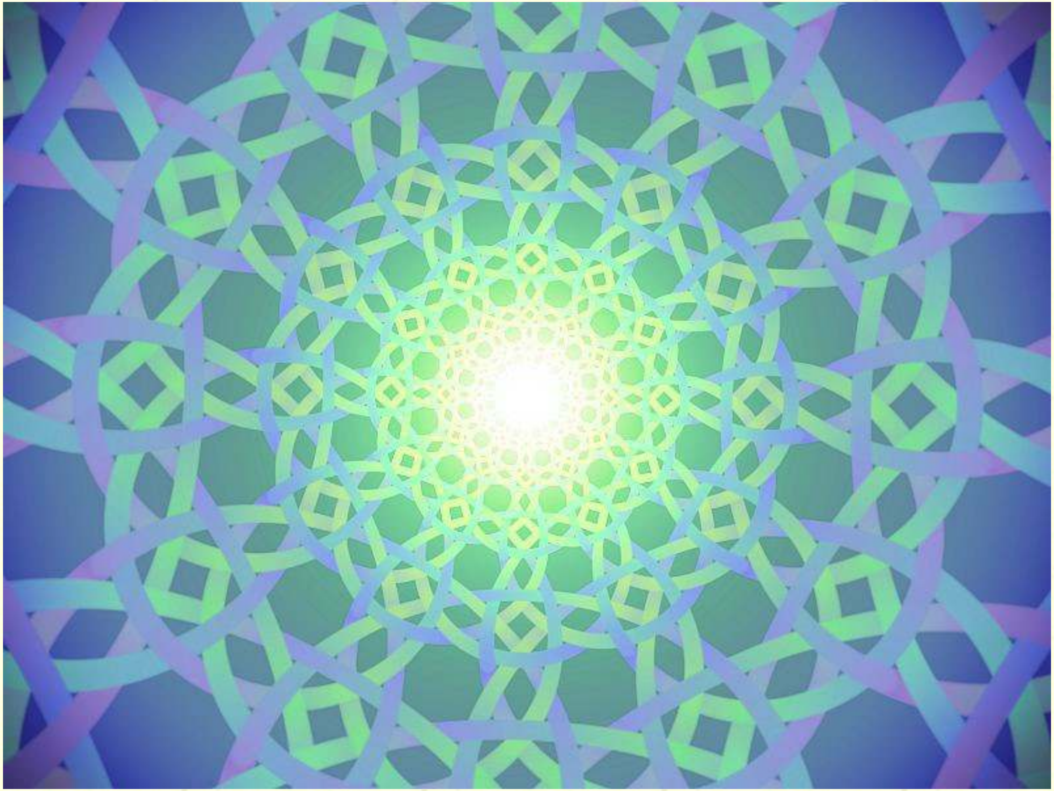
- Early writers on Fibonacci regarded Bonacci as a family name so that Fib-Bonacci is like the English names of Robin-son or John-son.

- Others think Bonacci may be a kind of nickname meaning "lucky son" , literally, "son of good fortune".

Fibonacci art

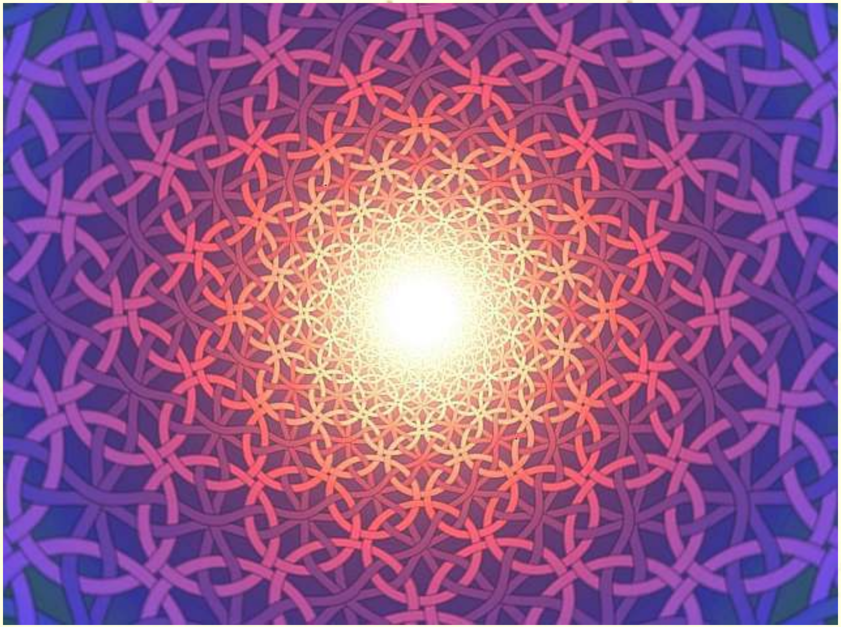
Ned May

Fibonacci art



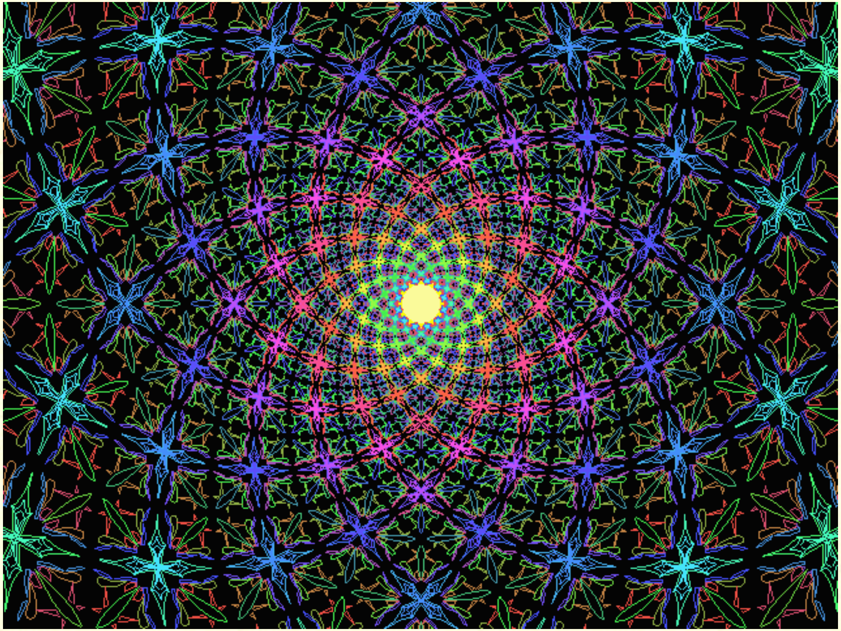
Ned May

Fibonacci art



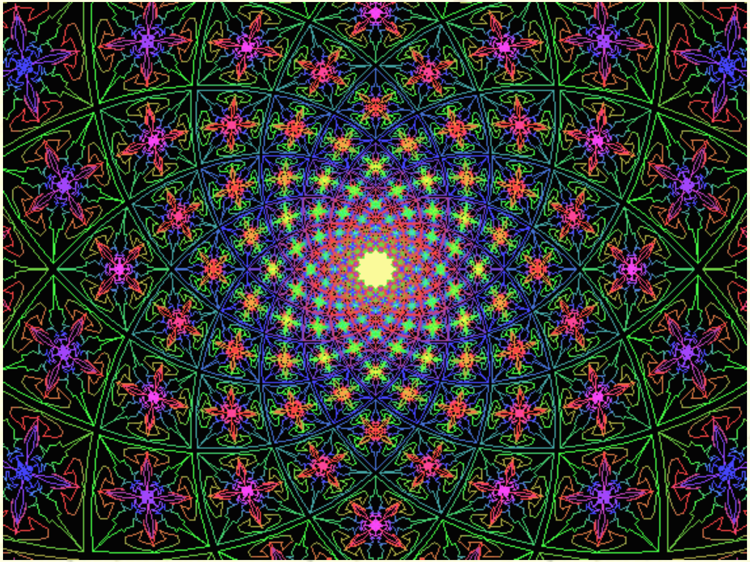
Ned May

Fibonacci art



Ned May

Fibonacci art



Ned May

Fibonacci art

Billie Ruth Sudduth

Fibonacci art

- The chimney of the Turku power station in Finland has the Fibonacci numbers on it in 2 m neon lights.
- It was the first commission of the Turku City Environmental Art Project in 1994.
- The artist, Mario Merz, calls it "Fibonacci Sequence 1-55" and says "it is a metaphor of the human quest for order and harmony among chaos."