

You must do things you think you cannot do.

Eleanor Roosevelt

School of the Art Institute of Chicago

Geometry of Art and Nature

Frank Timmes

ftimmes@artic.edu

flash.uchicago.edu/~fxt/class_pages/class_geom.shtml

Syllabus

1	Sept 03	Basics and Celtic Knots
2	Sept 10	Golden Ratio
3	Sept 17	Fibonacci and Phyllotaxis
4	Sept 24	Regular and Semiregular tilings
5	Oct 01	Irregular tilings
6	Oct 08	Rosette and Frieze groups
7	Oct 15	Wallpaper groups
8	Oct 22	Platonic solids
9	Oct 29	Archimedian solids
10	Nov 05	Non-Euclidean geometries
11	Nov 12	Bubbles
12	Dec 03	Fractals

Sites of the Week

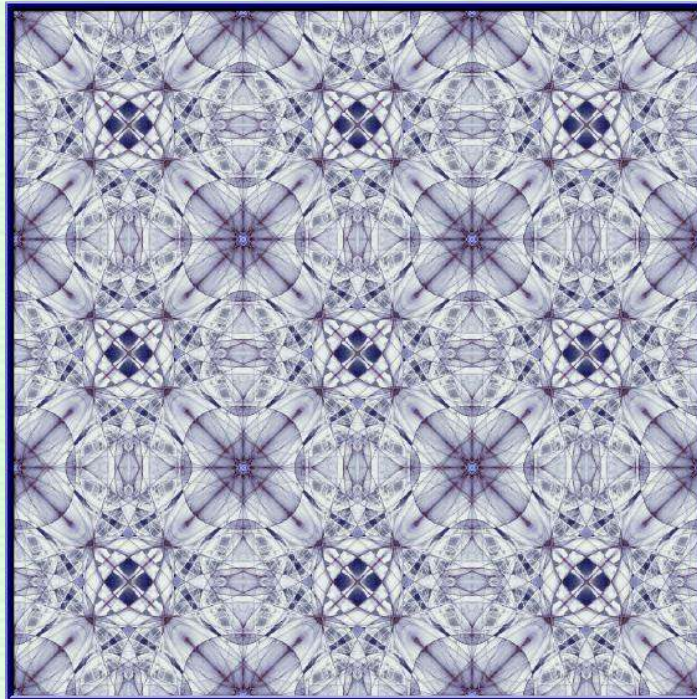
- www2.spsu.edu/math/tile/
- www.clarku.edu/~djoyce/wallpaper/seventeen.html
- hverrill.net/Wallpaper/wallindex.html
- nothung.math.uh.edu/~patterns/

Class #7

- Plane Symmetry
- There are only 17 wallpaper patterns

Wallpaper

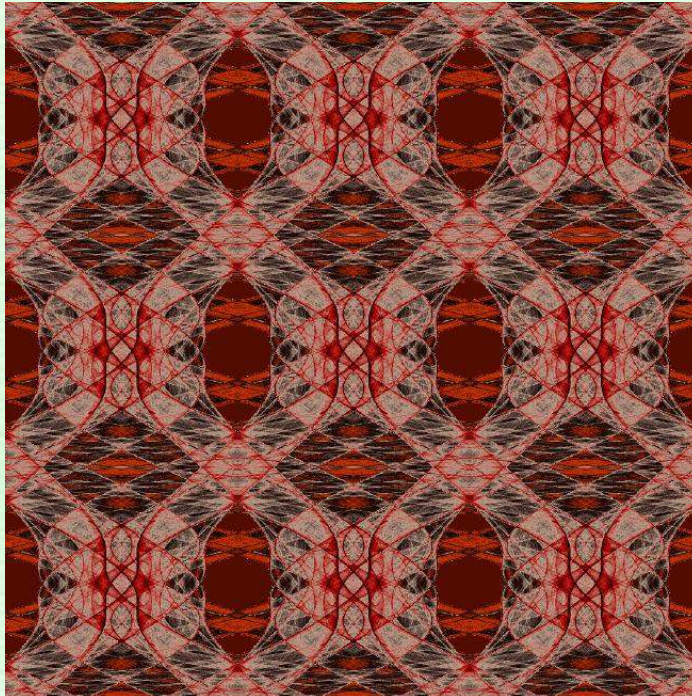
- A wallpaper pattern (also called a plane crystallographic group) is a pattern with translation symmetry in two directions.



p4m

Wallpaper

- They are essentially an arrangement of frieze patterns stacked upon one another to fill the plane.

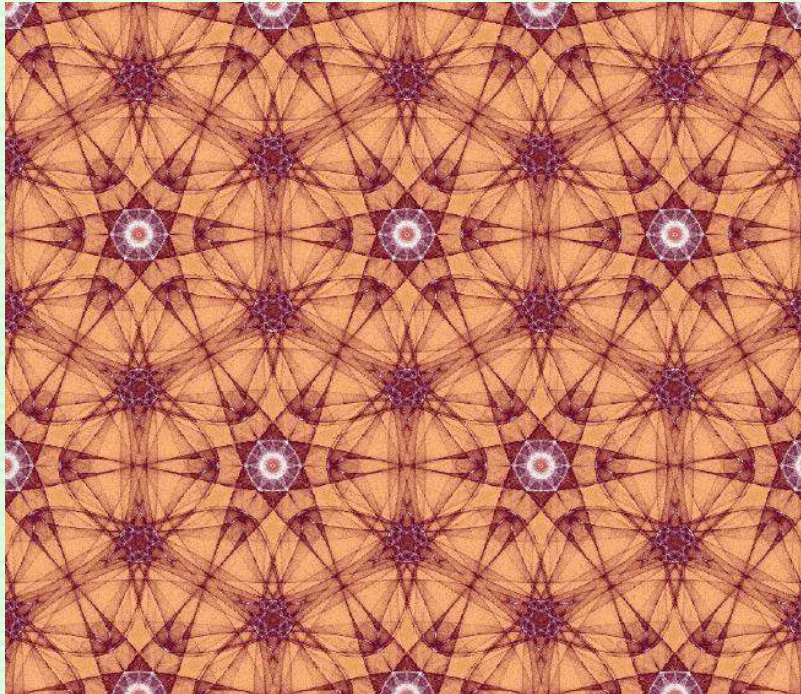


pmm

2000, Alberto Erazo

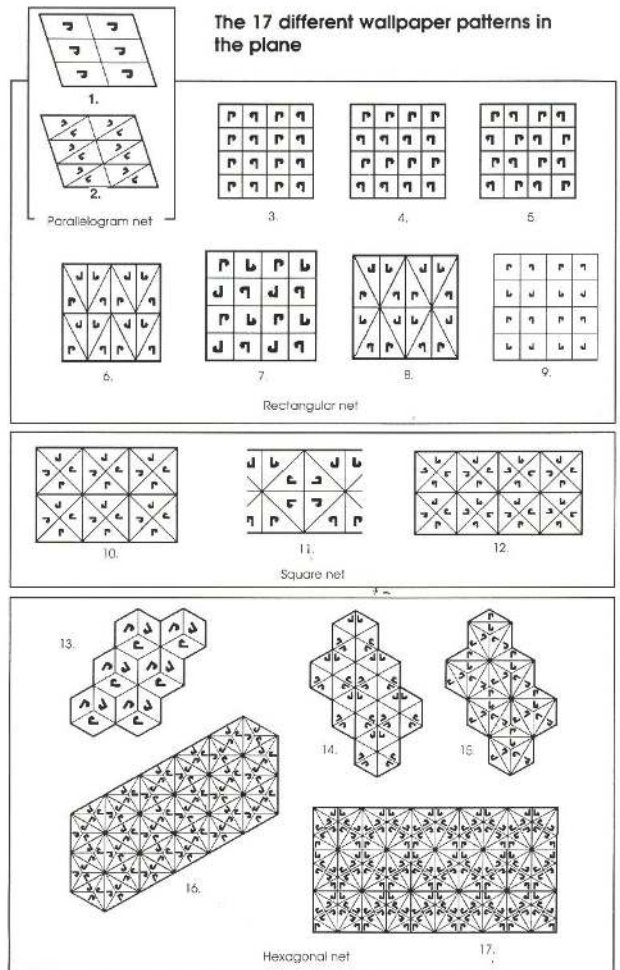
Wallpaper

- Any particular pattern is made from a combination of the four planar isometries: translation, rotation, reflection, and glide reflection.



p6m

- There are exactly 17 different wallpaper patterns.



Wallpaper

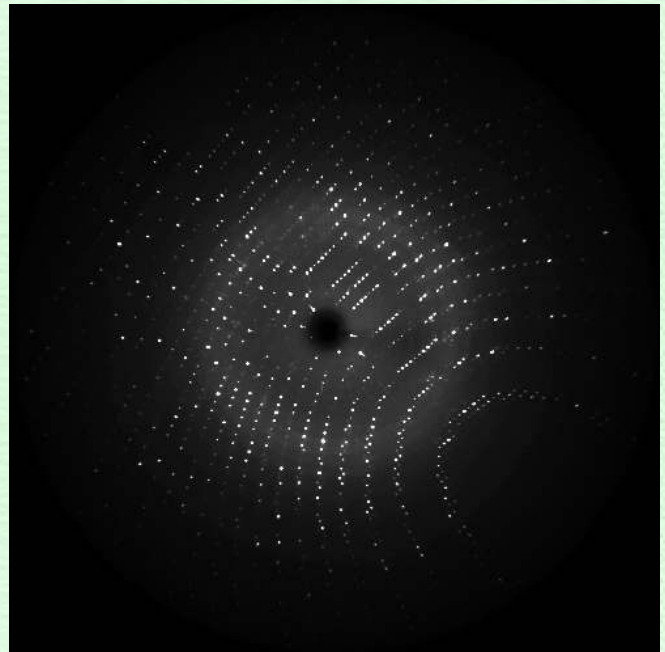
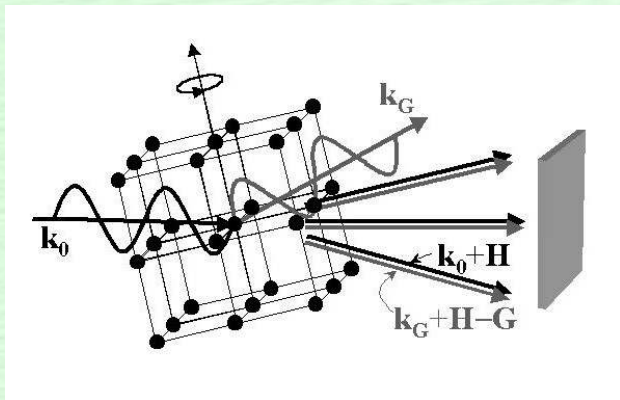
- I don't expect you to memorize these 17, but I do expect you to learn some of the principles that would allow you to classify them, or create designs with them.



Malahide wallpaper

Notation I

- The 17 wallpaper patterns are "named" by their crystallographic notation.



Notation I

p111

p211

p1m1

p1g1

c1m1

p2mm

p2mg

p2gg

c2mm

p411

p4mm

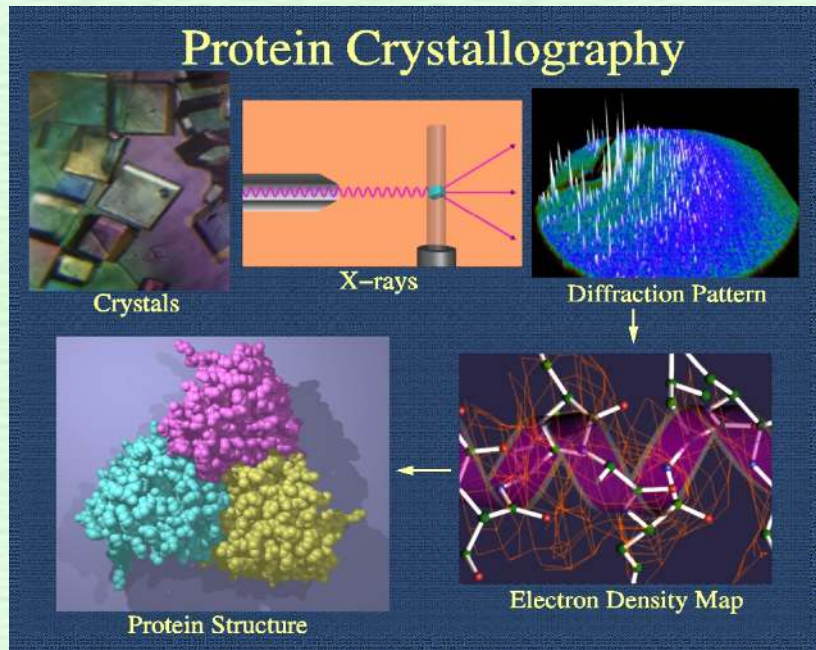
p4gm

p311

p3m1

p31m

- Each name is four characters long: _ _ _ _



Notation I

p111

p211

p1m1

p1g1

c1m1

p2mm

p2mg

p2gg

c2mm

p411

p4mm

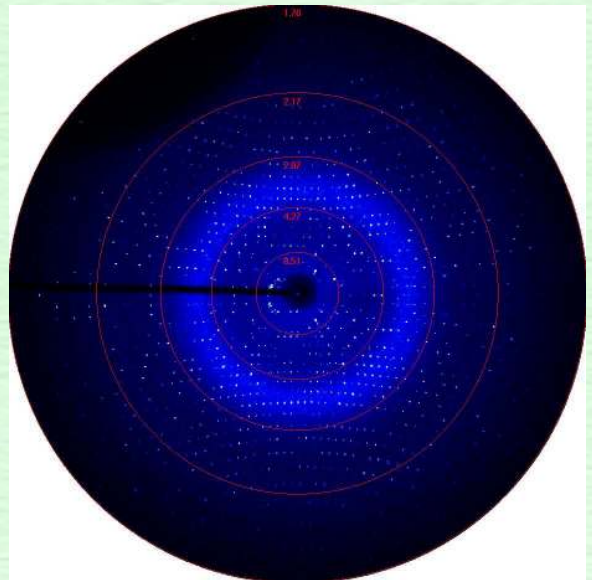
p4gm

p311

p3m1

p31m

- The first blank is filled by **p** or **c**, for the type of cell.
- Lattices of parallelograms, rectangles, squares, and hexagons have **p** (primitive), while a rhombic lattice gets **c** (centered).



Notation I

p111

p211

p1m1

p1g1

c1m1

p2mm

p2mg

p2gg

c2mm

p411

p4mm

p4gm

p311

p3m1

p31m

- The second blank is filled with the highest n-fold rotation. This could be 1 for no rotation, 2 for 2-fold (180°) rotation, 3 for a 3-fold (120°) rotation, 4 for a 4-fold (90°) rotation, or 6 for a 60° rotation.



Notation I

p111

p211

p1m1

p1g1

c1m1

p2mm

p2mg

p2gg

c2mm

p411

p4mm

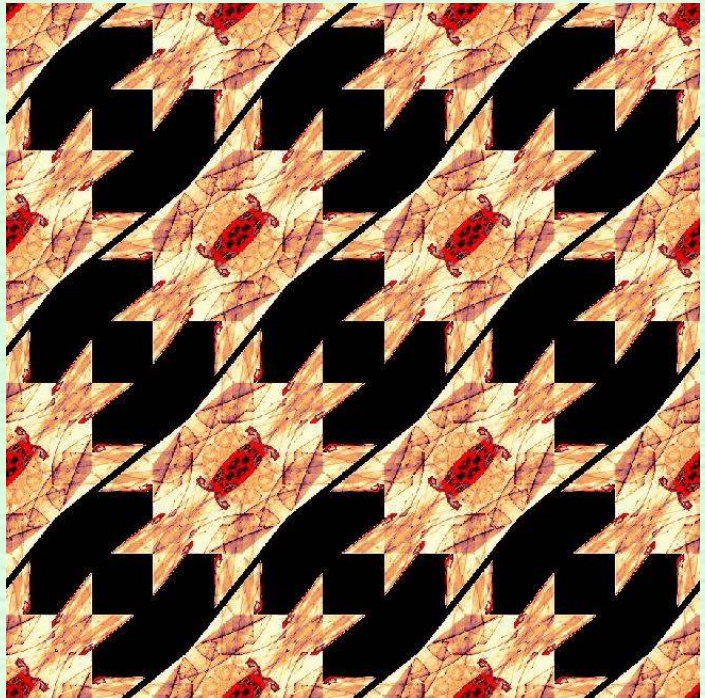
p4gm

p311

p3m1

p31m

- The third blank is filled with **m** if there is a vertical mirror, **g** if there is a glide reflection, and **1** otherwise.



Notation I

p111

p211

p1m1

p1g1

c1m1

p2mm

p2mg

p2gg

c2mm

p411

p4mm

p4gm

p311

p3m1

p31m

- The fourth blank is filled with **m** if there is a mirror at an angle consistent with the second blank, **g** if it is a glide reflection at an angle consistent with the second blank, and **1** otherwise.



Notation I

p111

p211

p1m1

p1g1

c1m1

p2mm

p2mg

p2gg

c2mm

p411

p4mm

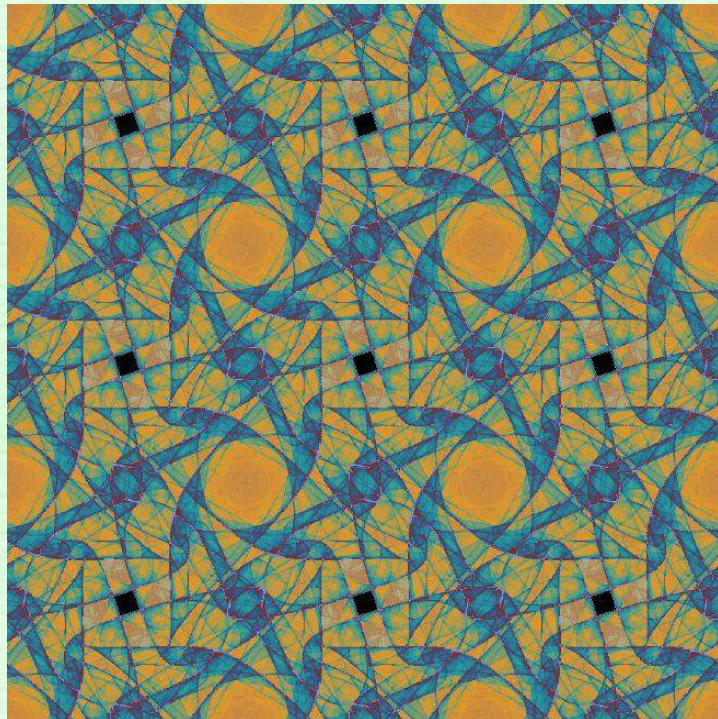
p4gm

p311

p3m1

p31m

- In all cases, 1 designates no to the question asked for each blank.



P1

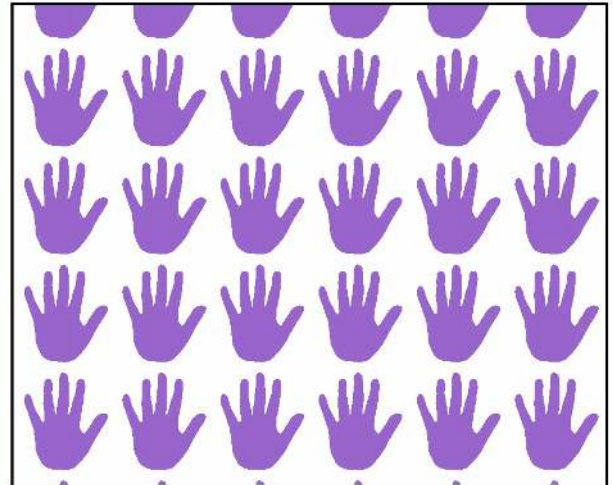
- This is the simplest symmetry group.
- It consists only of translations.
The two translation axes may be inclined at any angle to each other.
- Its base tile and lattice are any parallelogram.



Motif and
Base tile

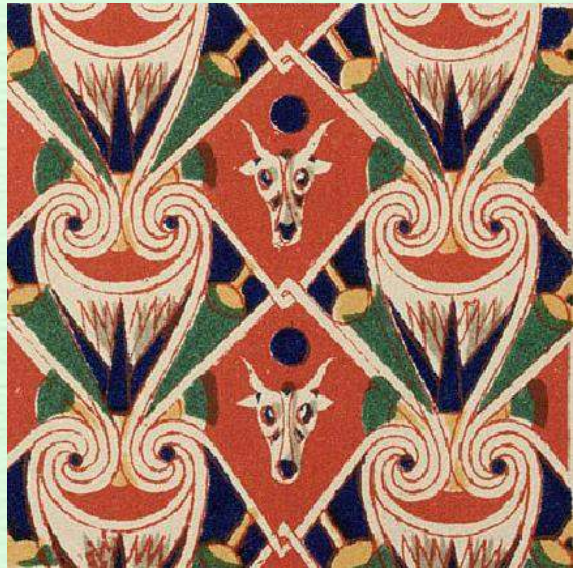


Lattice unit



P1 Wallpaper

- For many actual wallpaper patterns, translations are the only isometries that leave the pattern invariant.



Egyptian

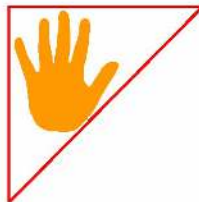
Stained Glass in Cathedral
of Bourges, Middle ages

- Usually, horizontal translations are not invariant on commercial wallpaper.
- Instead, the pattern is raised or lowered on adjacent wallpaper strips.



P2

- This group differs only from P1 in that it contains 180° rotations, that is, rotations of order 2, also called "half-turns."



Motif and
Base tile



Lattice unit



P2 Wallpaper

P2

- The two translation axes may be inclined at any angle to each other.
Its lattice is any parallelogram.



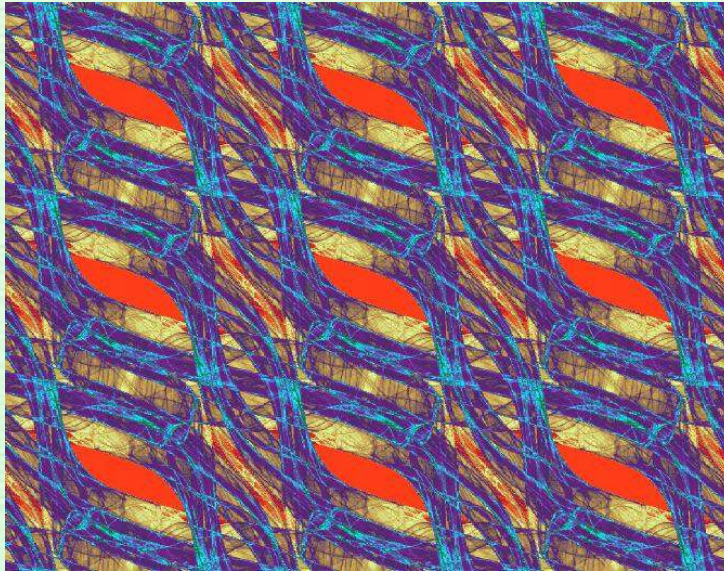
Egyptian



Egyptian

P2

- It may be a little difficult to see the half-turns; they're subtle.



P2
2000, Luke Grochowski

- In this example, as in all patterns with this symmetry, there are four essentially different centers of half-turns.

Long form	Short form	Lattice
p111	p1	Parallelogram
p211	p2	Parallelogram
p1m1	pm	Rectangle
p1g1	pg	Rectangle
c1m1	cm	Rhombus
p2mm	pmm	Rectangle
p2mg	pmg	Rectangle
p2gg	pgg	Rectangle
c2mm	cmm	Rhombus
p411	p4	Square
p4mm	p4m	Square
p4gm	p4g	Square
p311	p3	Hexagon
p3m1	p3m1	Hexagon
p31m	p31m	Hexagon

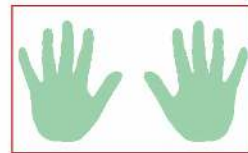
- Most of the time you'll see the short form of the wallpaper name rather than the long form.
- The short form of the name is derived by dropping any irrelevant or redundant characters from the long form.

PM

- This is our first group with reflections.

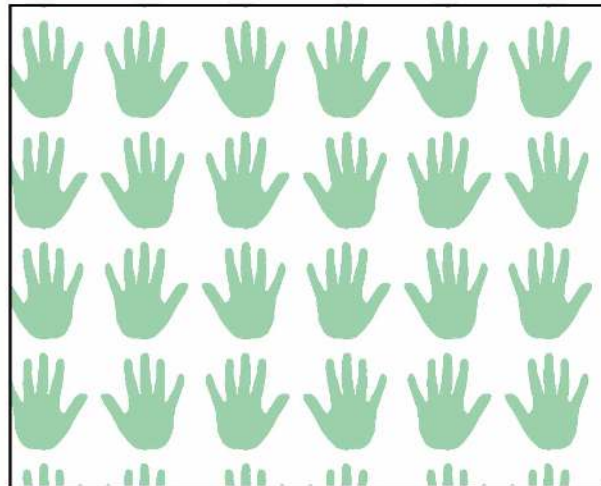


Motif and
Base tile



Lattice unit

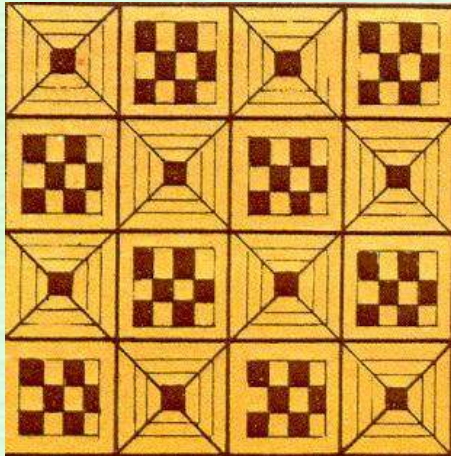
- The axes of reflection are parallel to one axis of translation and perpendicular to the other axis of translation.



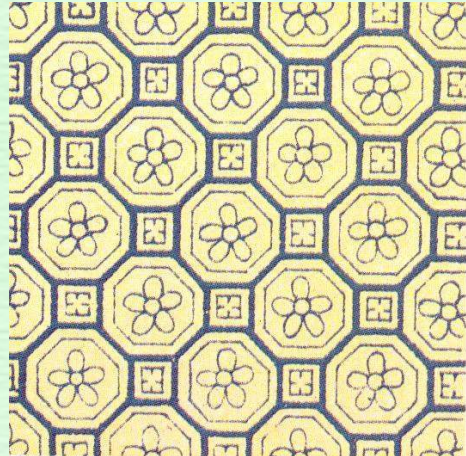
PM Wallpaper

PM

- The base tile is one-half the lattice unit.



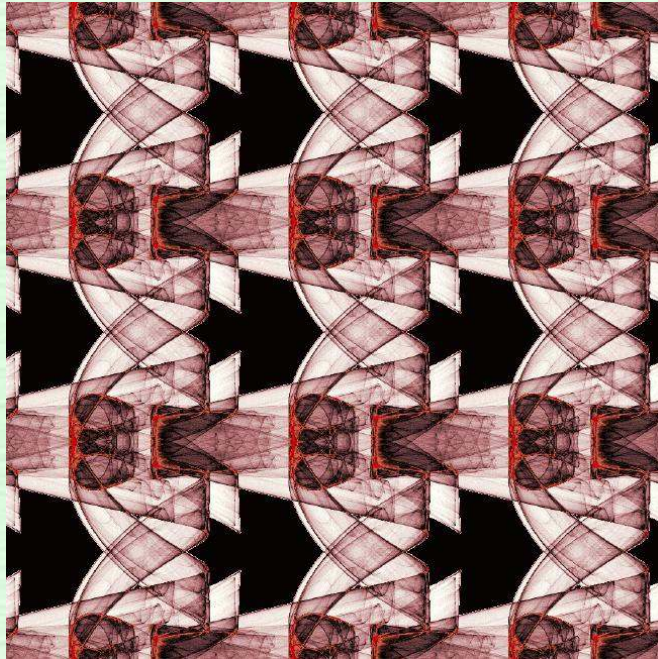
Hindu



Chinese

PM

- Reflections are usually easy to see; humans almost seem to be designed to recognize them, although it helps if the axis of reflection is vertical.



- This is the first group containing glide reflections.

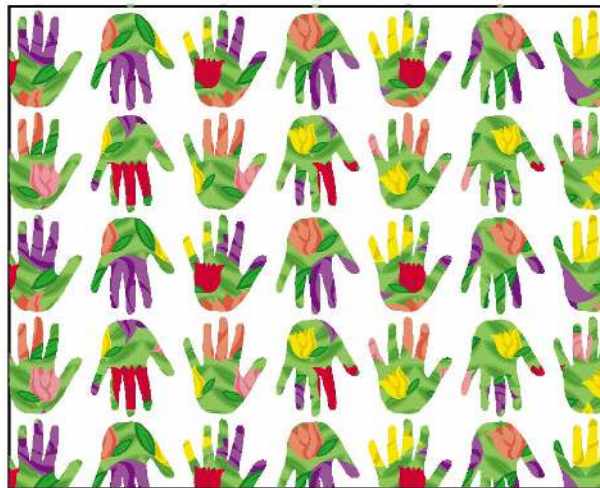


Motif and
Base tile



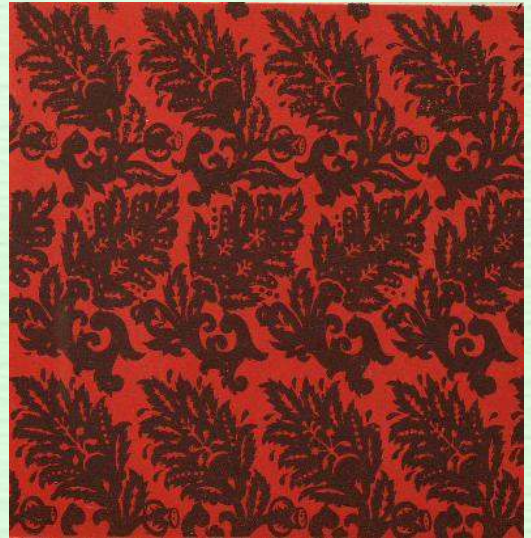
Lattice unit

- The axes of the glide reflection are parallel to one axis of translation and perpendicular to the other axis of translation.



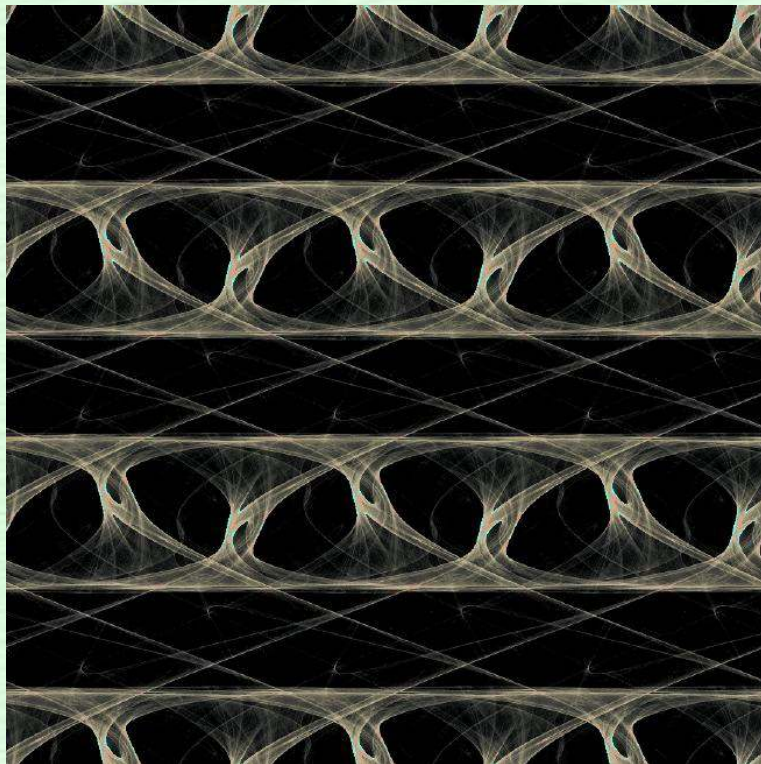
PG Wallpaper

- Half-turns were hard to see, but glide reflections can be even harder to see.

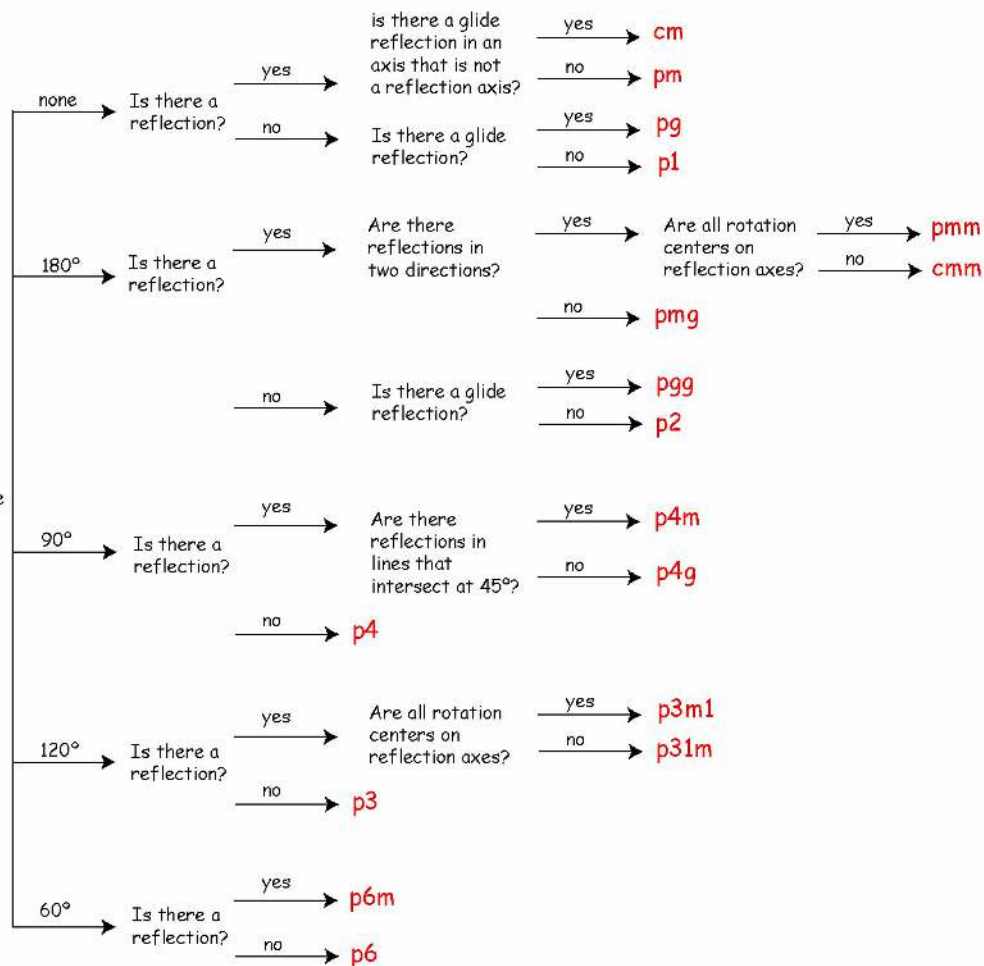


From a damask chair cover
in Kent, England; Elizabethan

PG



What is the
smallest
rotation?



CM

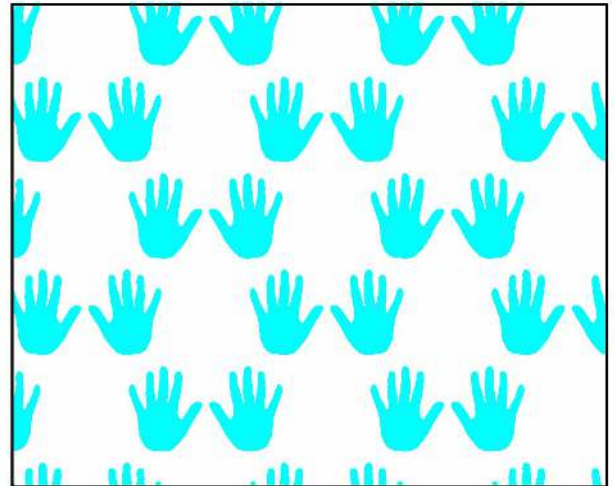
- This group contains reflections and glide-reflections with parallel axes.



Motif and
Base tile



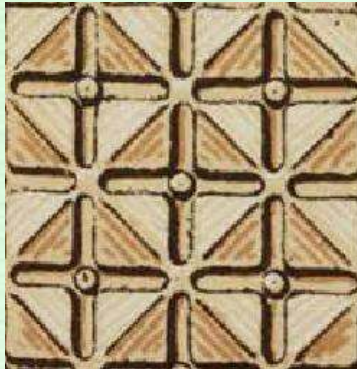
Lattice unit



CM Wallpaper

CM

- The translations may be inclined at any angle to each other, but the axes of the reflections bisect the angle formed by the translations, so the lattice for the translation group is a rhombus.



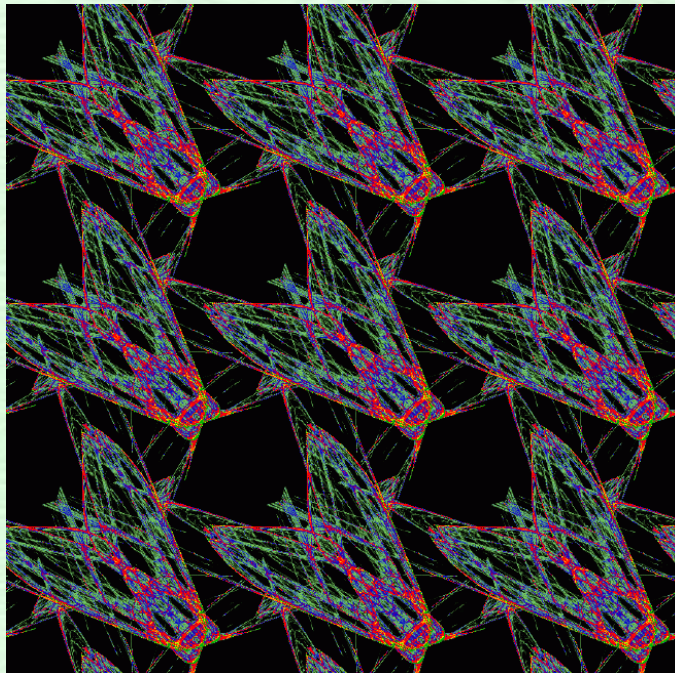
From St. Denis, France,
12th century, Byzantine



From a drapery in a tomb
at Westminster; Elizabethan

CM

- A base tile for the symmetry group is half the rhombus.



cm
2000, Mike Field
and John Nguyen

PMM

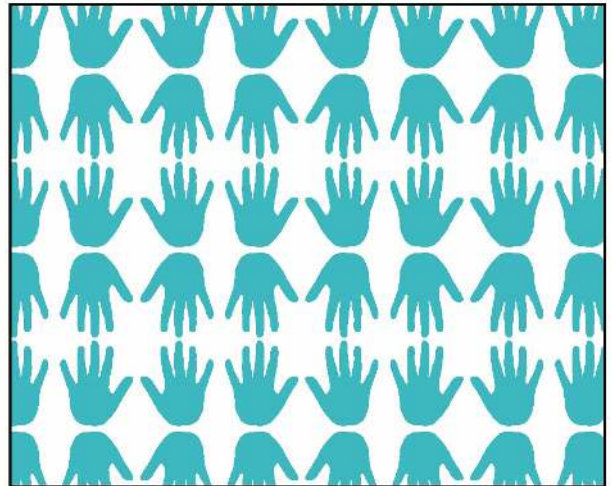
- This symmetry group contains reflections whose axes are perpendicular.
- There are no glide-reflections, except those which have the same axis as an axis of reflection.



Motif and
Base tile



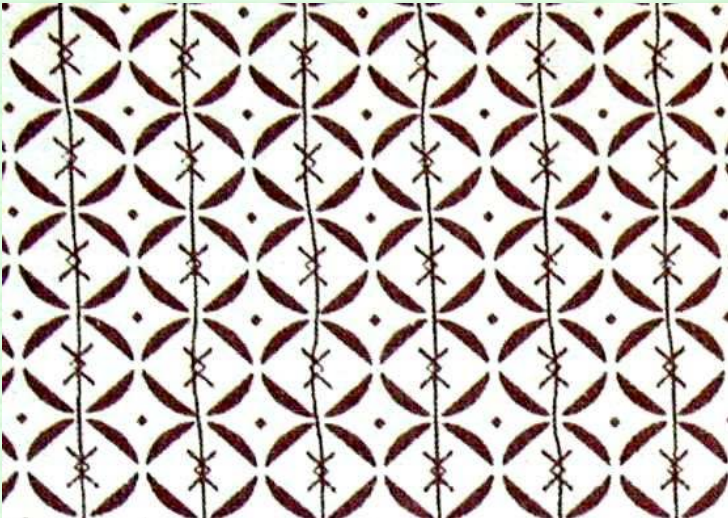
Lattice unit



PMM Wallpaper

PMM

- The only rotations are half-turns whose fixed points lie at intersections of axes of reflection.



Contemporary fabric



Egyptian

PMM

- The lattice is rectangular and the base tile is a quarter-rectangle.



Long form	Short form	Lattice
p111	p1	Parallelogram
p211	p2	Parallelogram
p1m1	pm	Rectangle
p1g1	pg	Rectangle
c1m1	cm	Rhombus
p2mm	pmm	Rectangle
p2mg	pmg	Rectangle
p2gg	pgg	Rectangle
c2mm	cmm	Rhombus
p411	p4	Square
p4mm	p4m	Square
p4gm	p4g	Square
p311	p3	Hexagon
p3m1	p3m1	Hexagon
p31m	p31m	Hexagon

PMG

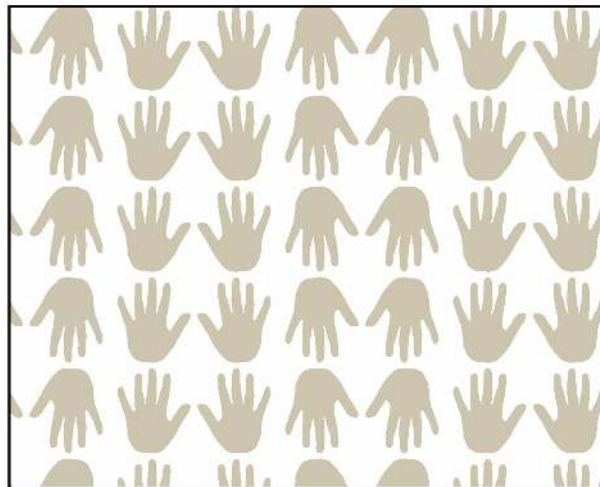
- This group has both reflections (with parallel axes) and half-turns.
- The fixed points of the half-turns do not lie on the axes of reflection.



Motif and
Base tile



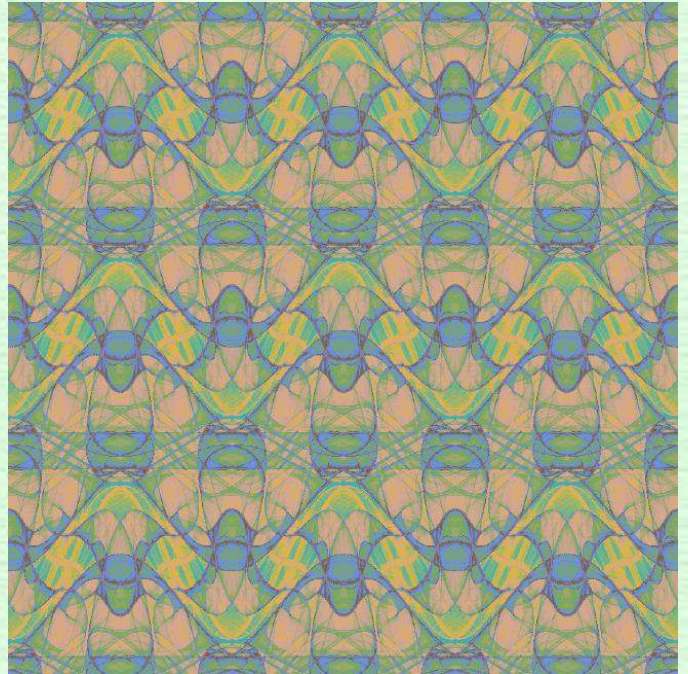
Lattice unit



PMG Wallpaper

PMG

- The lattice is rectangular, and the base tile is a quarter-rectangle.



pmg
2000, Diane La Franca

PGG



Motif and
Base tile



Lattice unit

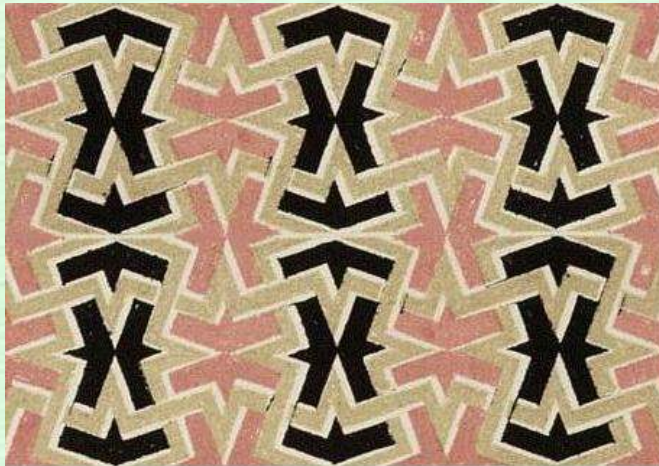
- This group contains no reflections, but it has glide-reflections and half-turns.



PGG Wallpaper

PGG

- There are perpendicular axes for the glide-reflections, and the fixed points of the half-turns do not lie on these axes.



Arabic

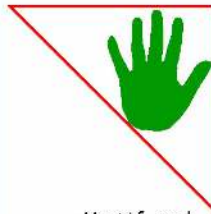
PGG

- The lattice is rectangular, and the base tile is a quarter-rectangle.



CMM

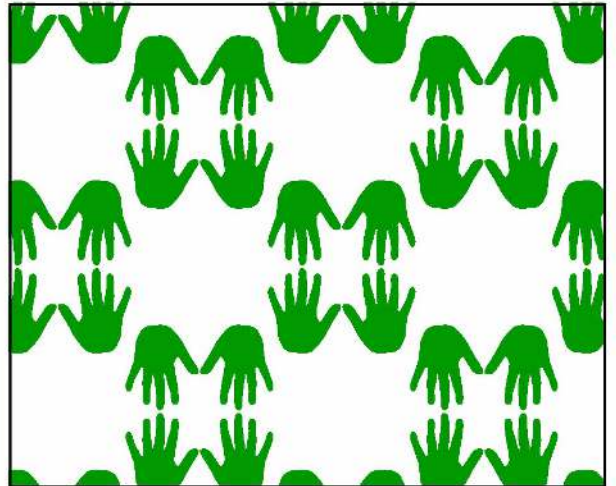
- This group has perpendicular reflection axes and half-turns.



Motif and
Base tile



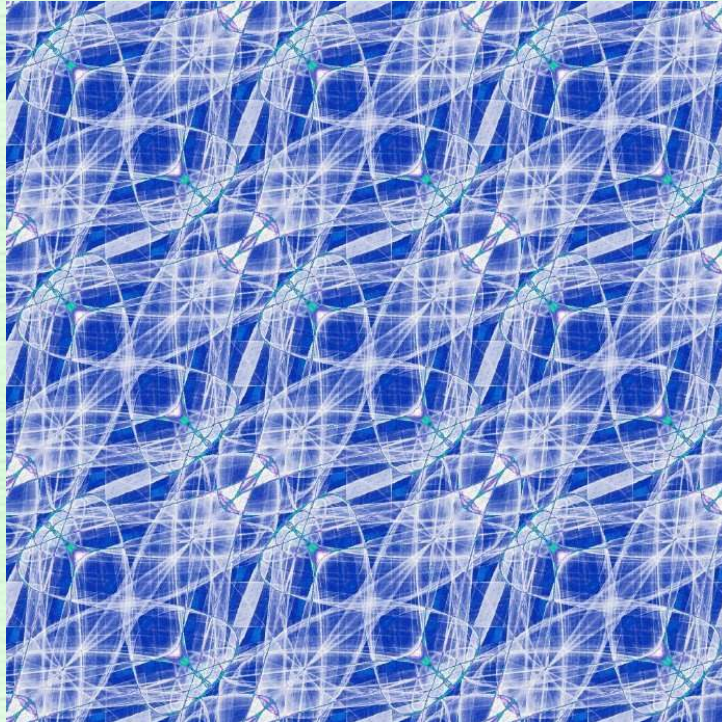
Lattice unit



CMM Wallpaper

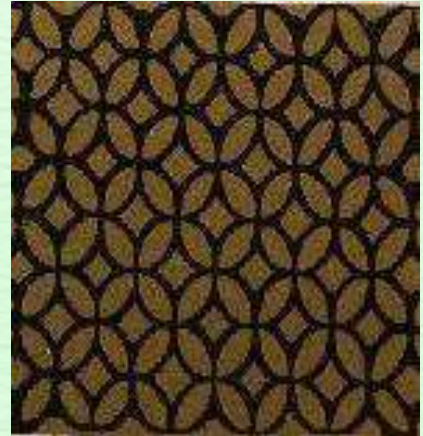
CMM

- Some centers of the half-turns lie at intersections of the axes of reflection, but others lie at intersections of the axes of the glide reflections.



CMM

- The lattice is rhombic, and the base tile is a quarter of the rhombus.

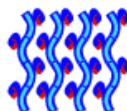


Indian

p1



pg



pgg



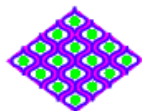
pm



cm



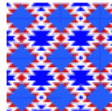
cmm



pmg



pmm



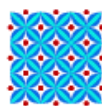
p2



p4



p4m



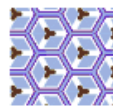
p4g



p3



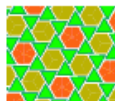
p3ml



p31m



p6



p6m



P4

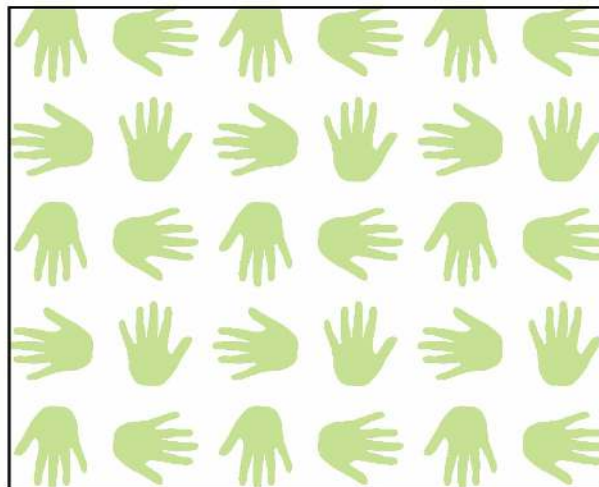
- This is the first group with a 90° rotation, that is, a rotation of order 4. It also has rotations of order 2, that is, half-turns.



Motif and
Base tile



Lattice unit



P4 Wallpaper

P4

- The centers of the half-turns are midway between the centers of the order-4 rotations.



Moorish

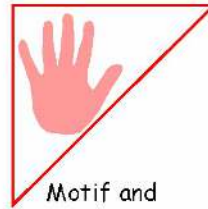
P4

- The lattice is square, and a quarter of a square is the base tile.

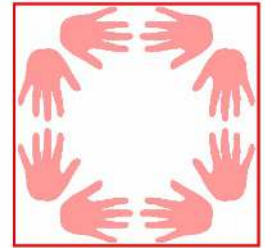


P4M

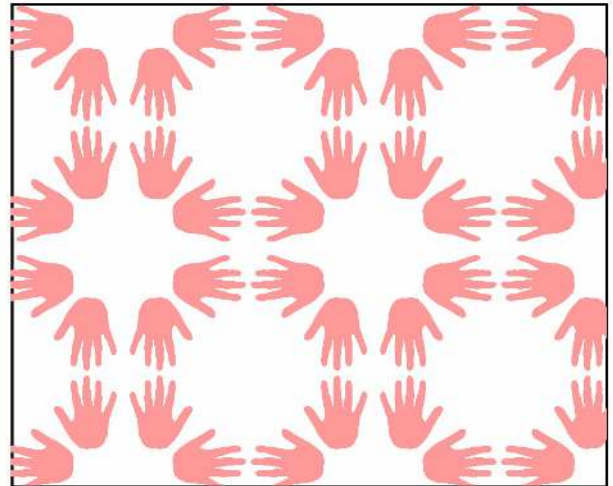
- This group differs from p4 in that it also has reflections.



Motif and
Base tile



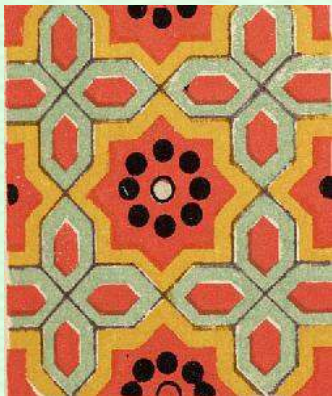
Lattice unit



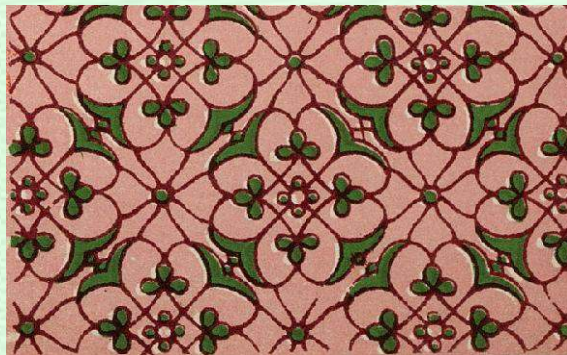
P4M Wallpaper

P4M

- The axes of reflection are inclined to each other by 45° so that four axes of reflection pass through the centers of the order-4 rotations.



Persian



Persian

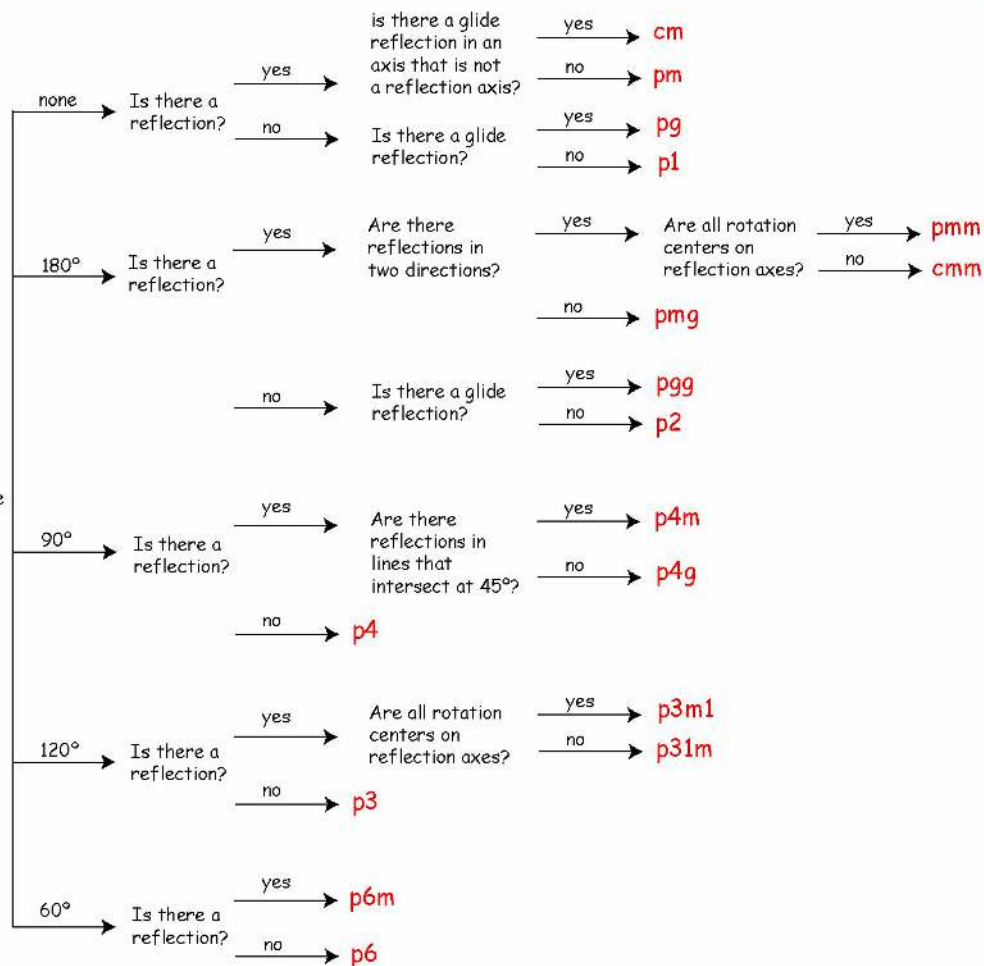
P4M

- The lattice is square, and a triangular eighth of the square is the base tile.



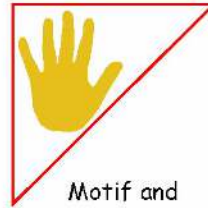
Patio tiles at
Pensacola Beach, Florida

What is the
smallest
rotation?



P4G

- Like p4m, this group also contains reflections and rotations of orders 2 and 4.
- But the axes of reflection are perpendicular (none at 45°), and none of the centers of the rotations of order 4 lie on the reflection axes.



Motif and
Base tile



Lattice unit



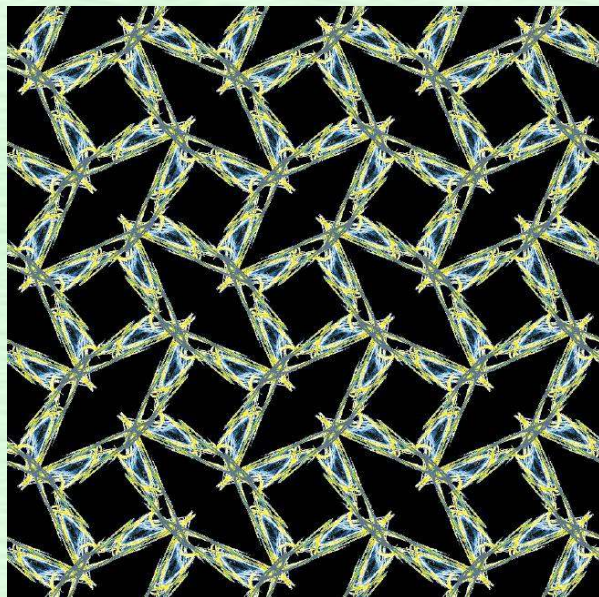
P4G Wallpaper

P4G

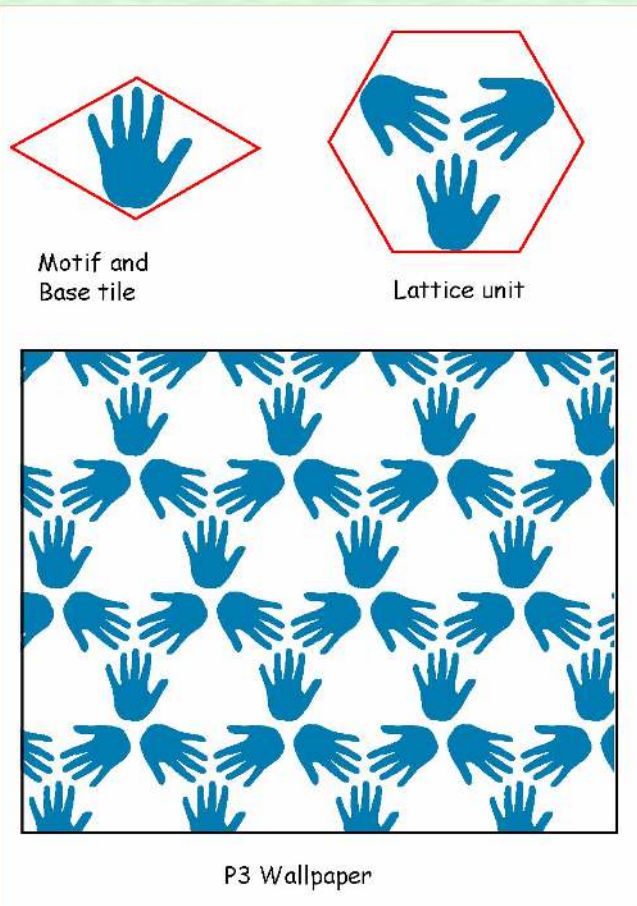
- The lattice is a square, and a triangular eighth of the square is the base tile.



Chinese

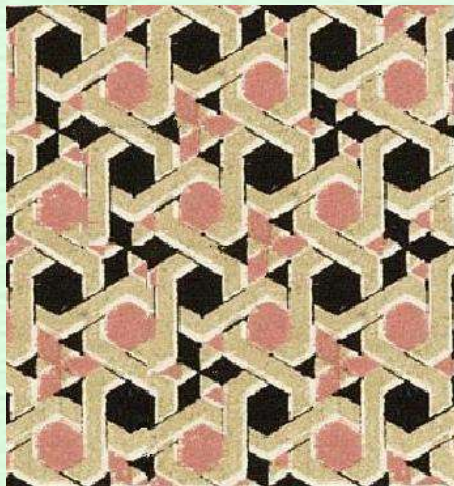


- This is the first group that contains a 120° rotation, that is, a rotation of order 3, and the first one whose lattice is hexagonal.

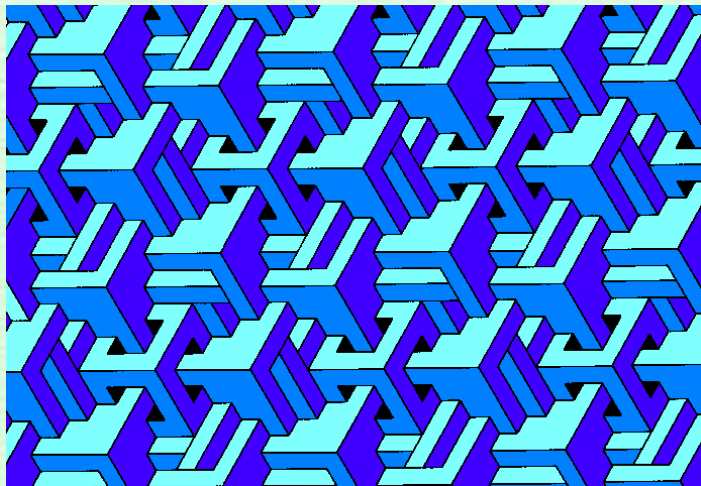


P3

- The base tile is a rhombus, one third of the hexagon lattice.



Arab



p3
1981, John Locke

P3M1

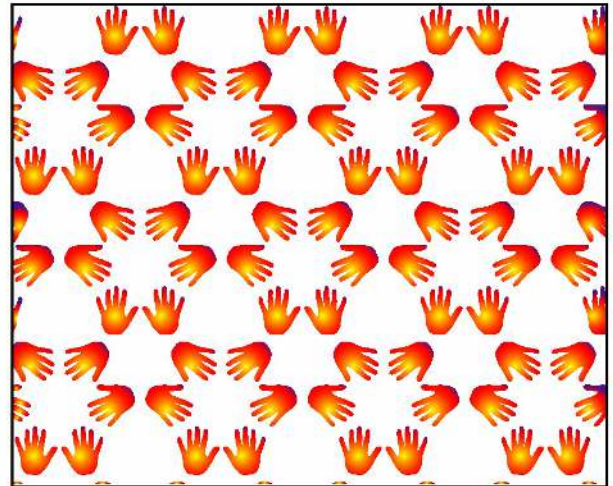
- This group contains reflections whose axes are inclined at 60° to one another and rotations of order 3.



Motif and
Base tile



Lattice unit



P3M1 Wallpaper

P3M1

- Some of the centers of rotation lie on the reflection axes, and some do not.



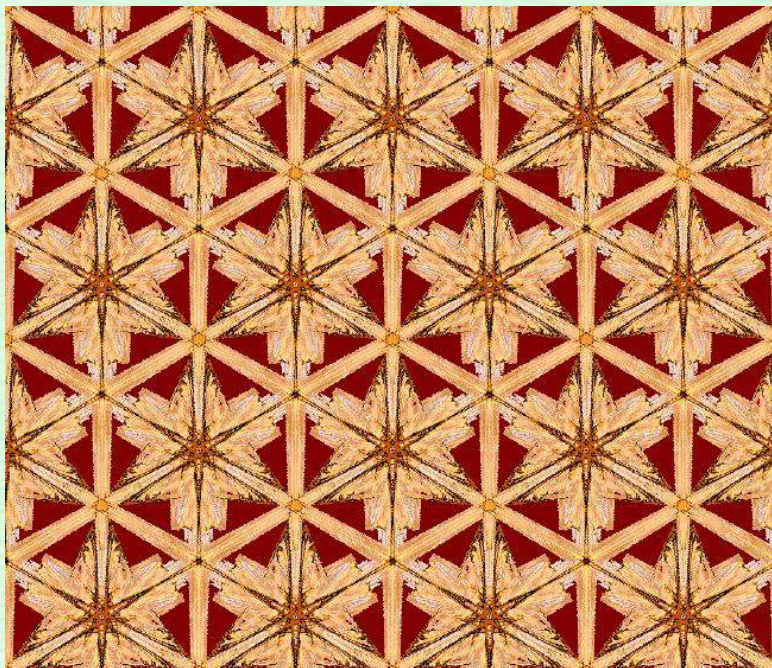
Chinese



Persian

P3M1

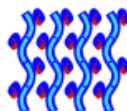
- The lattice is hexagonal, and the base tile is an equilateral triangle, one sixth of the hexagon.



p1



pg



pgg



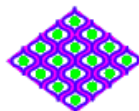
pm



cm



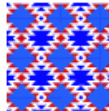
cmm



pmg



pmm



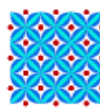
p2



p4



p4m



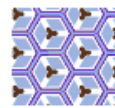
p4g



p3



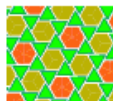
p3ml



p31m



p6

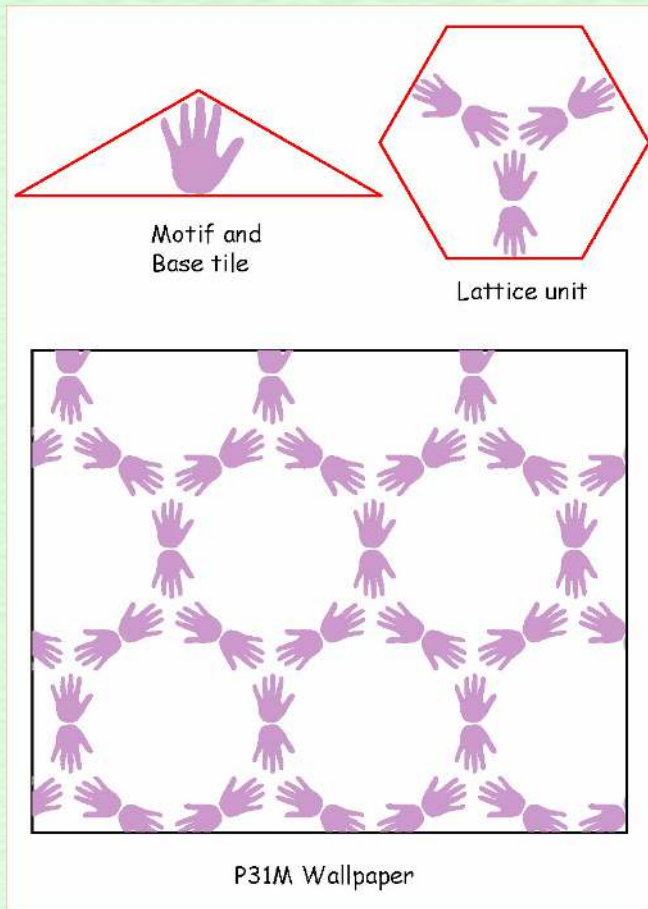


p6m



P31M

- This group is similar to $p3m1$ in that it contains reflections and order-3 rotations.
- The axes of the reflections are again inclined at 60° to one another, but for this group all of the centers of rotation lie on the reflection axes.



P31M

- The lattice is hexagonal, and the base tile is a $30^\circ - 30^\circ - 120^\circ$ triangle.

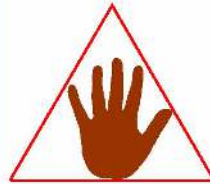


Chinese



Chinese

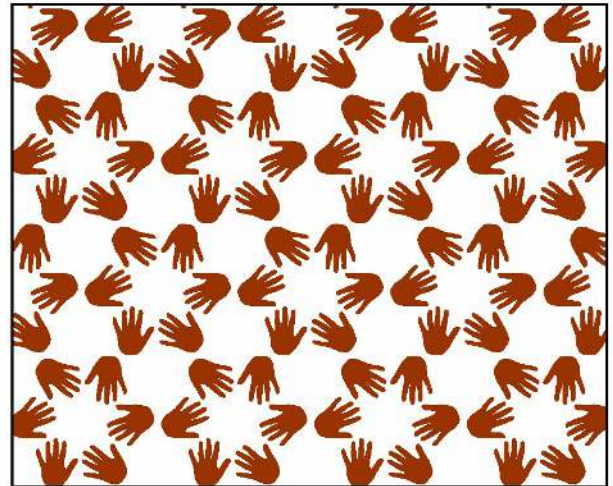
- This group contains 60° rotations, and rotations of orders 2 and 3, but neither reflections nor glide reflections.



Motif and
Base tile



Lattice unit

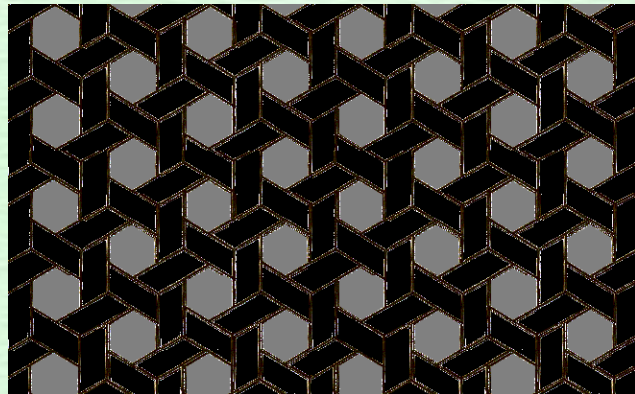


P6 Wallpaper

- The lattice is hexagonal, with the base tile being an equilateral triangle.

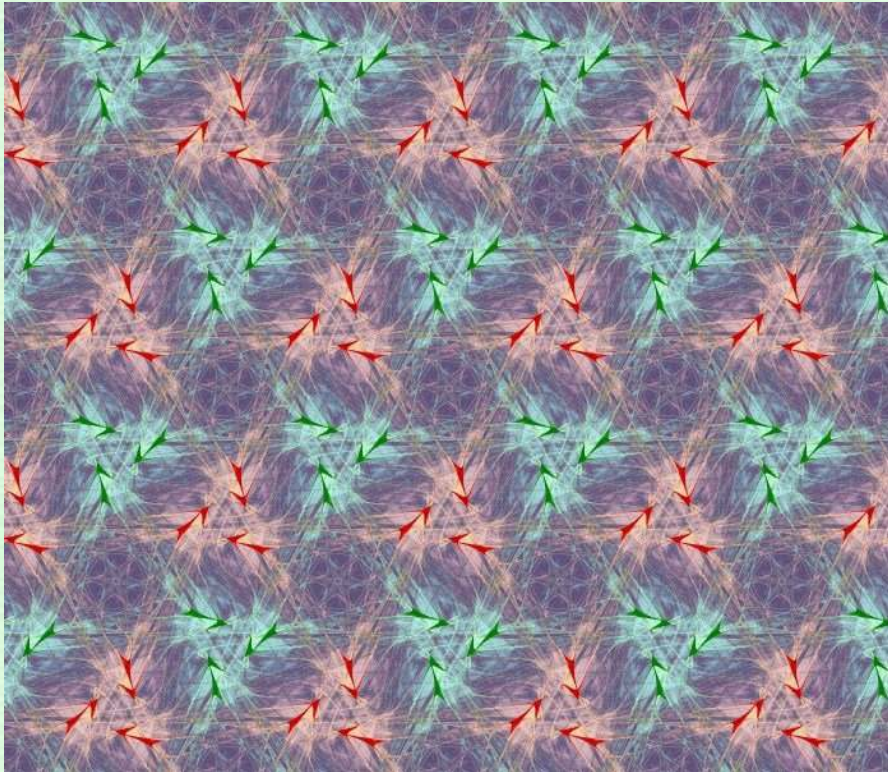


Moorish



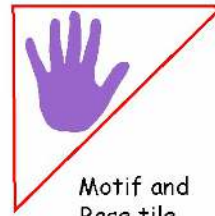
Persian

P6

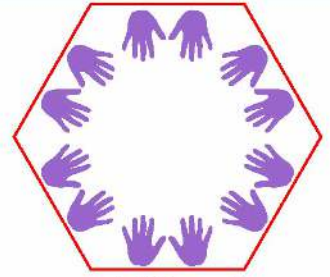


P6M

- This most complicated group has rotations of order 2, 3, and 6 as well as reflections.

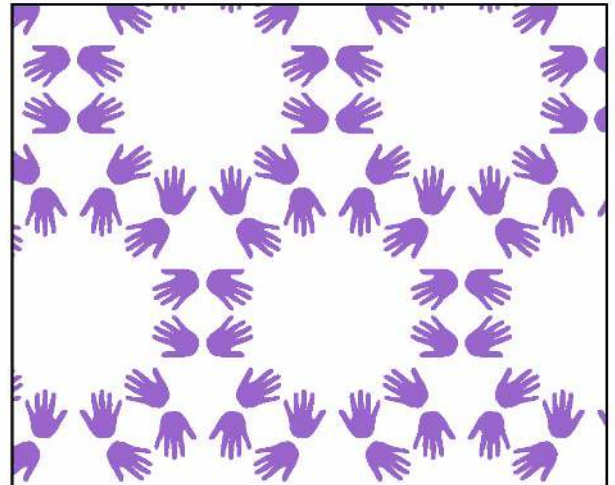


Motif and
Base tile



Lattice unit

- The axes of reflection meet at all the centers of rotation.



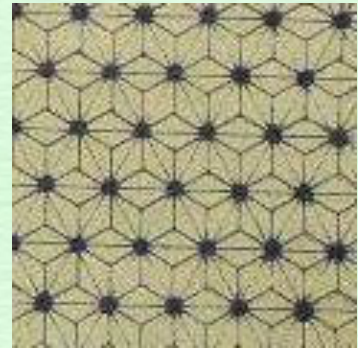
P6M Wallpaper

P6M

- At the centers of the order-6 rotations, six reflection axes meet and are inclined at 30° to one another.



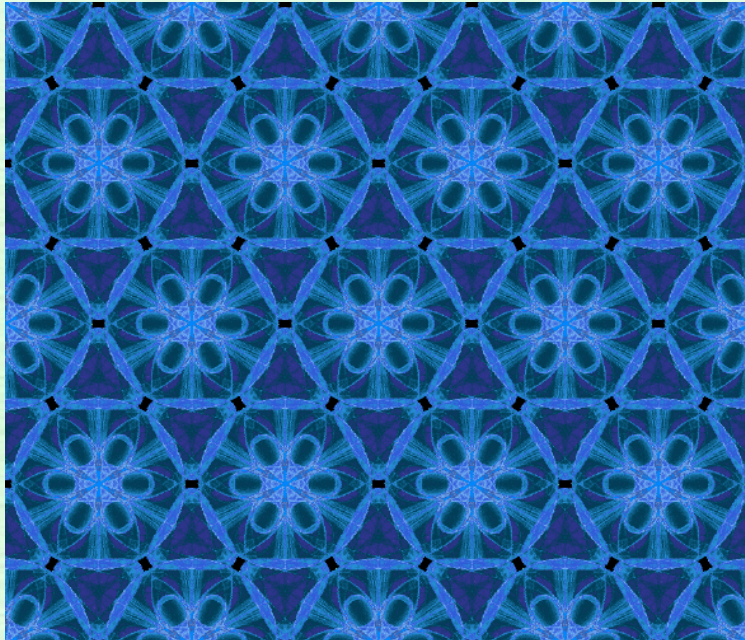
Byzantine



Chinese

P6M

- The lattice is hexagonal and the base tile is a $30^\circ - 60^\circ - 90^\circ$ triangle, one twelfth of a hexagon.





1



2



3



4



5



7



6



8



9



10



11



12



13



14



15



16



17



17 WALLPAPER GROUPS

In today's in-class construction, you'll play with the lattice patterns that fascinated a young M.C. Escher.