Geometry is the art of correct reasoning on incorrect figures.

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Geometry of
Art and Nature

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flash.uchicago.edu/~fxt/class_pages/class_geom.shtml

Syllabus

1	Sept 03	Basics and Celtic Knots
2	Sept 10	Golden Ratio
3	Sept 17	Fibonacci and Phyllotaxis
4	Sept 24	Regular and Semiregular tilings
5	Oct 01	Irregular tilings
6	Oct 08	Rosette and Frieze groups
7	Oct 15	Wallpaper groups
8	Oct 22	Platonic solids
9	Oct 29	Archimedian solids
10	Nov 05	Non-Euclidean geometries
11	Nov 12	Bubbles
12	Dec 03	Fractals

Sites of the Week

- www.geom.umn.edu/graphics/pix/Special_Topics/Hyperbolic_Geometry/
- members.tripod.com/professor_tom/hyperbolic/
- members.shaw.ca/quadibloc/maps/mapint.htm

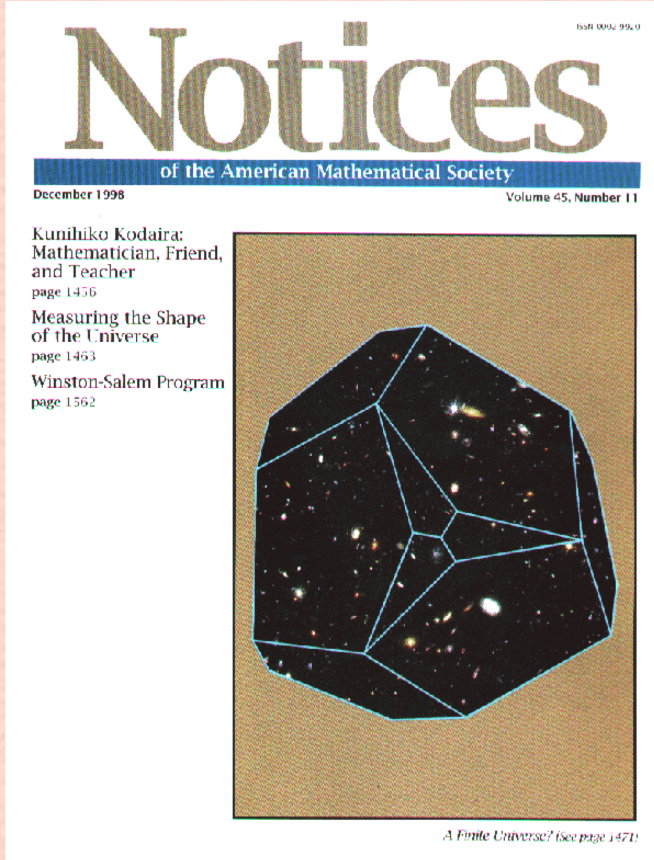
Class #10

- Big Bang cosmology
- Non-Euclidean spaces
- Hyperbolic and elliptical surfaces

Cosmology

- What is the shape of the universe?

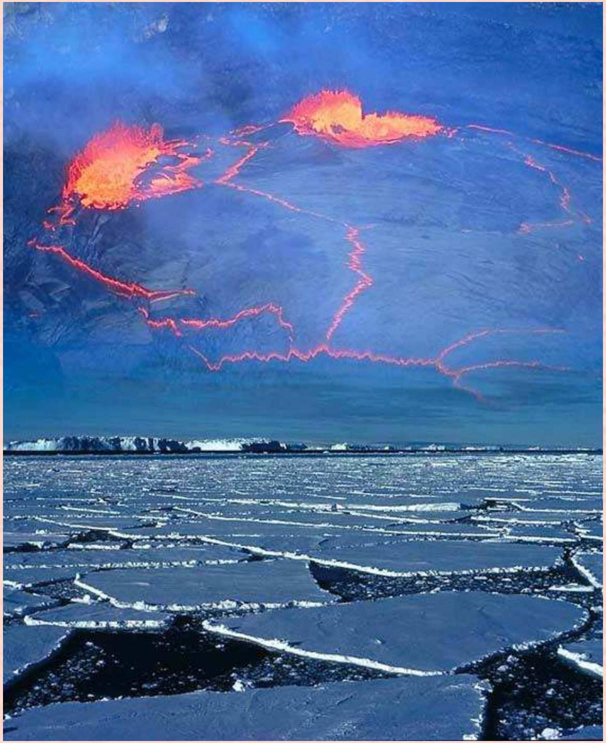
- This is an intriguing question, and one that carries with it other implications - like what the final fate of the universe will be.



Fire or Ice?

Some say the world will end in fire,
some say in ice.
From what I've tasted of desire
I hold with those who favor fire.
But if I had to perish twice,
I think I know enough of hate
to say that for destruction, ice
is also great.
And would suffice.

Robert Frost

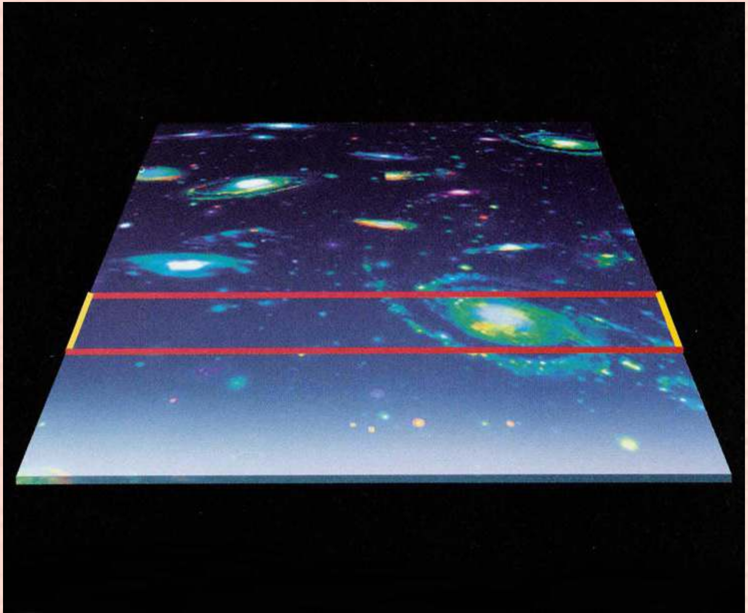


2002, Gregory G. and
Mary Beth Dimijian

Cosmology

- There are only three possibilities for the geometry (or curvature) of the universe when gravity is the only force to consider.

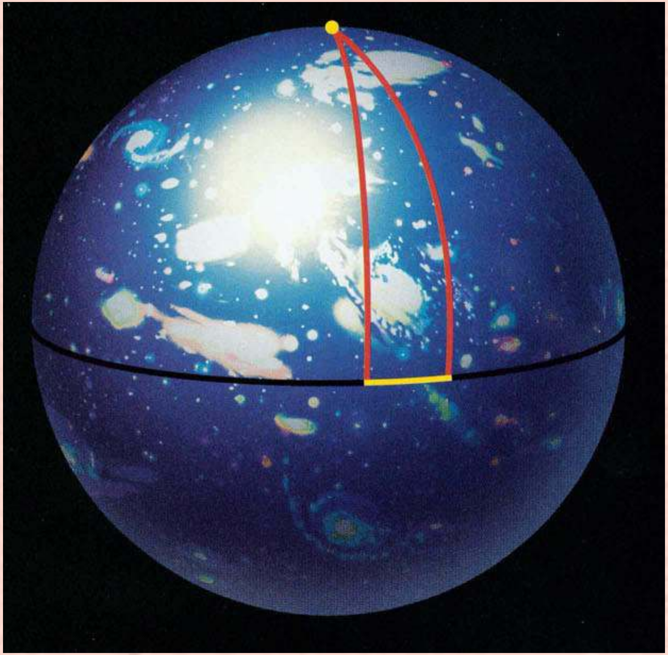
- It could have zero curvature and be flat, like a piece of paper.



- This is the geometry we've been discovering in this class.

Cosmology

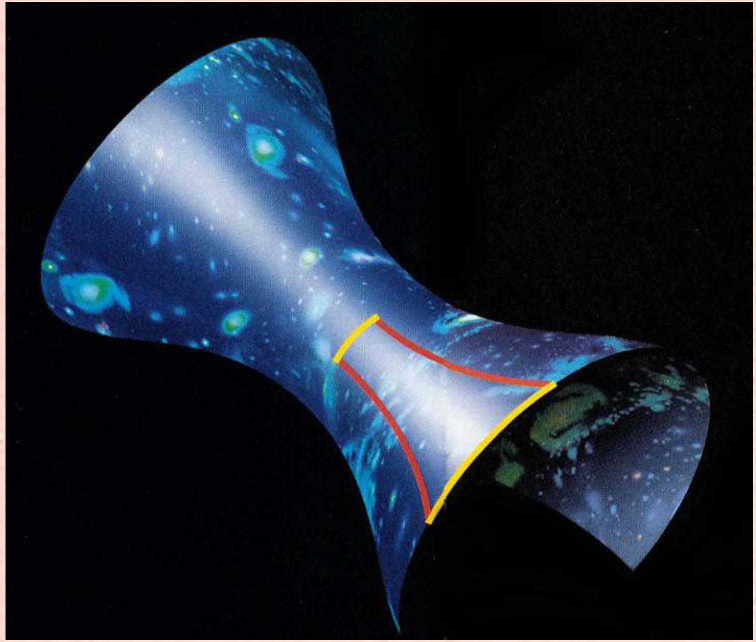
- The universe could have a positive curvature and have a geometry like the surface of a sphere.



- On a sphere, no matter which way you go, it curves the same way, so we call it positive curvature.

Cosmology

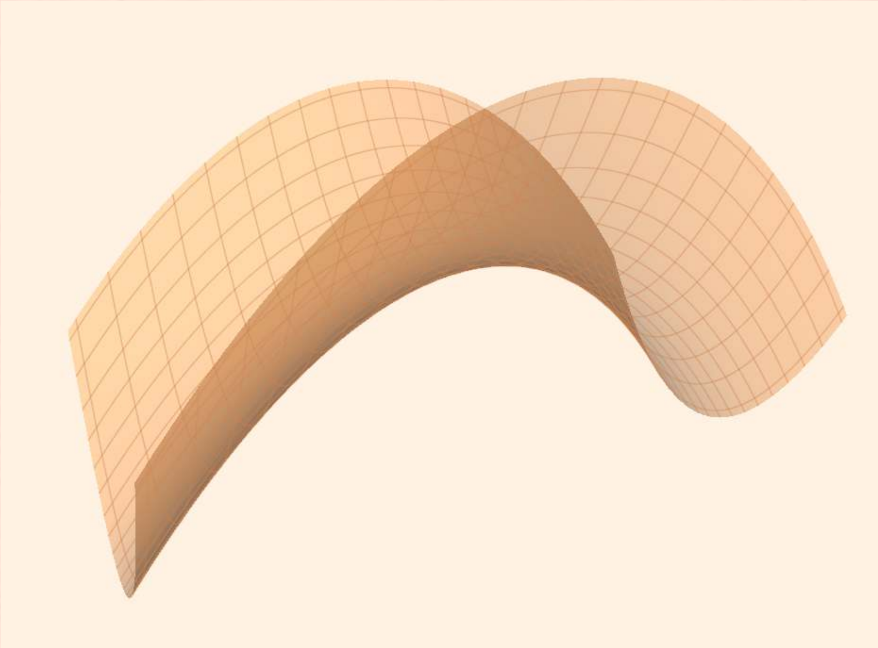
- The universe could have a negative curvature and have a geometry like the surface of a Pringle's potato chip.



- If you start at the front and move to the back, the surface curves down and then up.
- If you move from side to side, it curves up and then down.

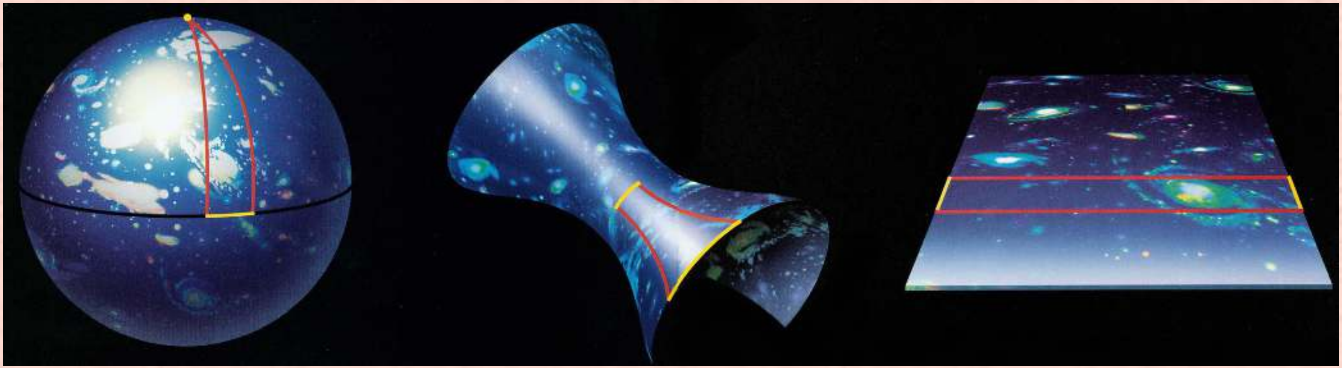
Cosmology

- This is the definition of negative curvature; opposite curvatures going in different directions.



Cosmology

- The curvature of the universe depends on how much mass is in the universe, and the amount of mass determines its ultimate fate.



- So, it is more than just a passing geometric fancy that we might want to measure the universe's curvature. Let's take a deeper look at these geometries.

Non-Euclidean

- The geometry of Euclid, the geometry of zero curvature that we've been exploring in this class, is based on a number of postulates.

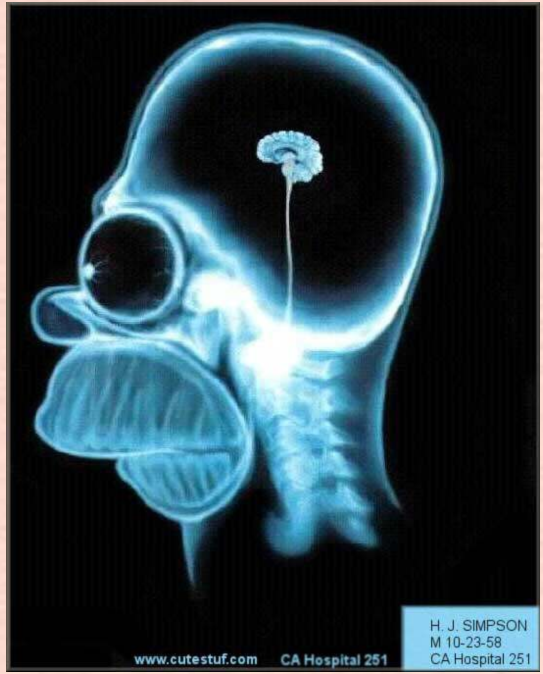
- A postulate is an obvious, or self-evident truth, that cannot be derived from other postulates.

Non-Euclidean

- The postulates of Euclidean geometry are:

1) You can draw a straight line between any two points.
2) You can extend any segment indefinitely.
3) You can draw a circle with any given point as a center and any given radius.
4) All right angles are equal.
5) Through a given point not on a given line
   there passes one line parallel to the given line.

Non-Euclidean

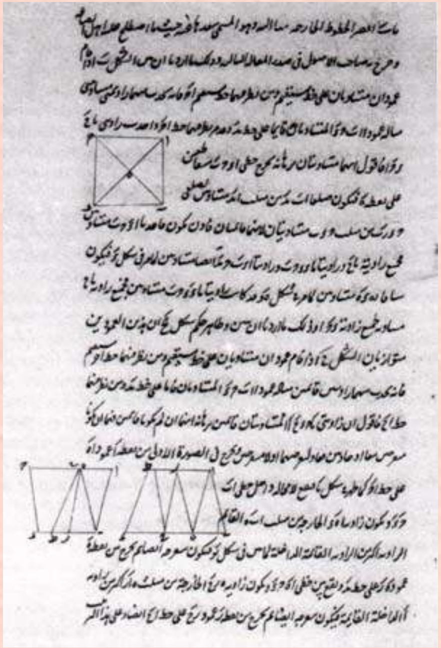


- The first four seem straightforward.

- The fifth postulate, the Parallel Postulate, sounds complicated. Even Euclid expressed some uneasiness about it, and delayed using it until Theorem 29.

Non-Euclidean

- Because the Parallel Postulate doesn't sound as obvious as the others, people tried for centuries to prove that it could be deduced from the others. They failed.

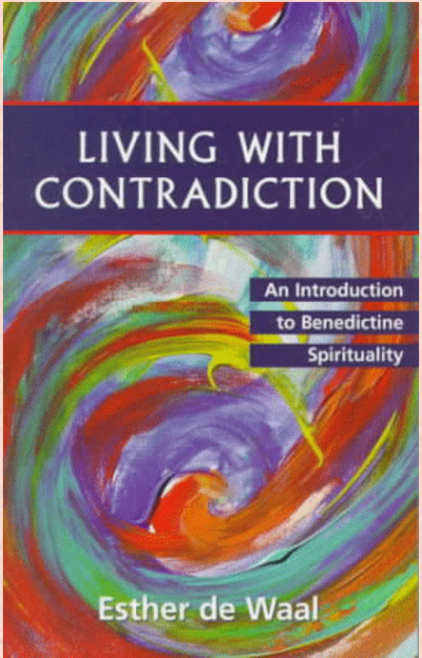


Euclid's parallels postulate received much attention from Islamic mathematicians throughout the history of medieval Arabic science. Nasir ad-Din at-Tusi's was probably the most mature treatment of the problem in Arabic.

Non-Euclidean

- Attempting an indirect proof, mathematicians in the late 18th century began assuming the fifth postulate was false and tried to reach a logical contradiction.

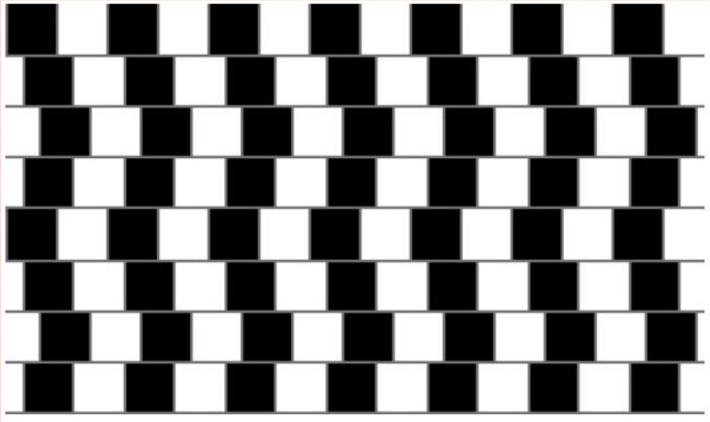
- There are two possible negations of the Parallel Postulate.



Non-Euclidean

1. Through any *given* point not on a *given* line there pass more than one line parallel to the *given* line.

2. Through a *given* point not on a *given* line there pass no lines parallel to the *given* line.



There are only perfect squares and parallel straight lines in this image.

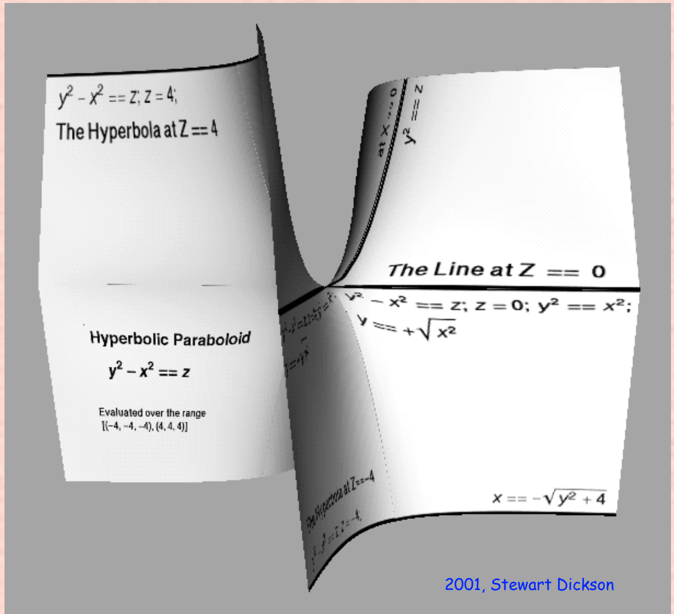
Non-Euclidean

- Carl Gauss (German, 1755-1855),
  Nikolai Lobachevsky (Russian, 1792-1856), and
  Janos Bolyai (Hungarian, 1802-1860)
  investigated the first assumption of more than one parallel line.

Non-Euclidean

- They discovered that this assumption led to a new geometry that didn't contradict any of Euclid's other postulates or any of the theorems that didn't depend on the Parallel Postulate.

- This non-Euclidean geometry is called hyperbolic geometry, "hyperbolic" from the Greek word for excessive (more than one parallel line).



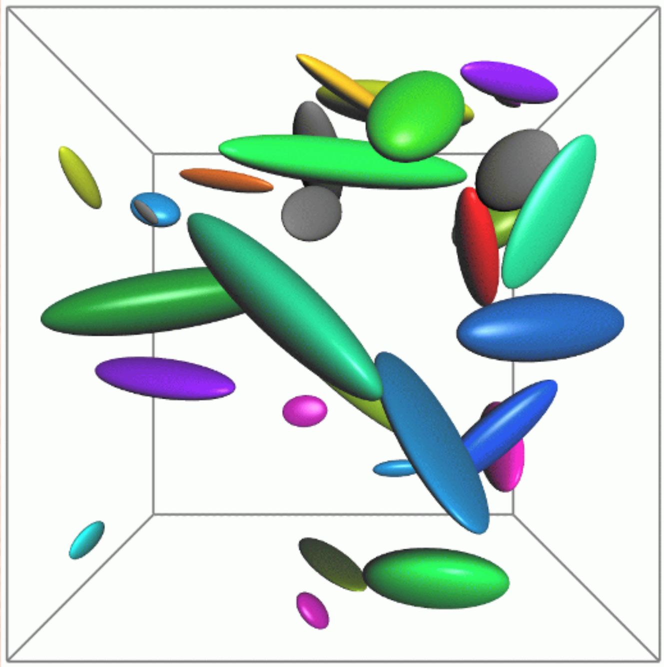
2001, Stewart Dickson

Non-Euclidean

- George Riemann (German, 1826-1866) investigated the second assumption of no parallel lines.

Non-Euclidean

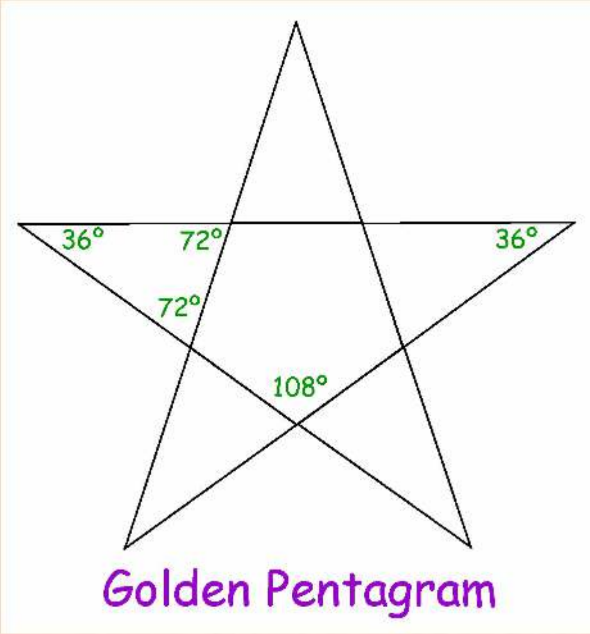
- He discovered that this assumption again led to a new geometry that didn't contradict any of Euclid's other postulates or any of the theorems that didn't depend on the Parallel Postulate.



- This non-Euclidean geometry is called elliptic geometry, "elliptic" from the Greek word for deficient (no parallel line).

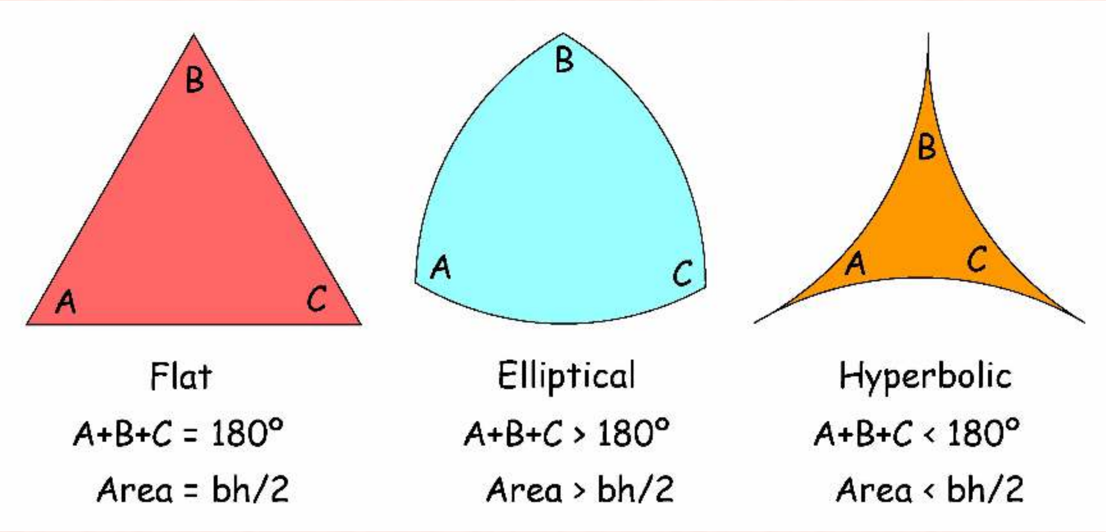
Non-Euclidean

- An example of a Euclidean theorem that relies on the Parallel Postulate is the Triangle Sum Theorem: the sum of the angles in a triangle is 180°.



Non-Euclidean

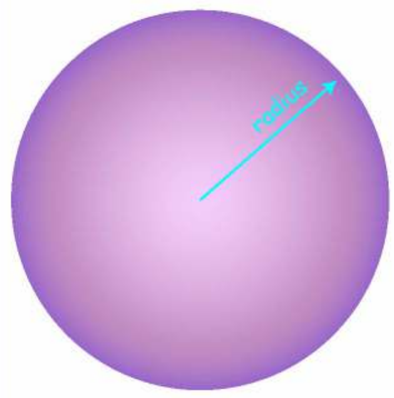
- In hyperbolic spaces the sum of the three angles is less than 180°, while in elliptical spaces the sum is more than 180°.



Non-Euclidean

- A circle is defined to be the set of all points equidistant from a given center.

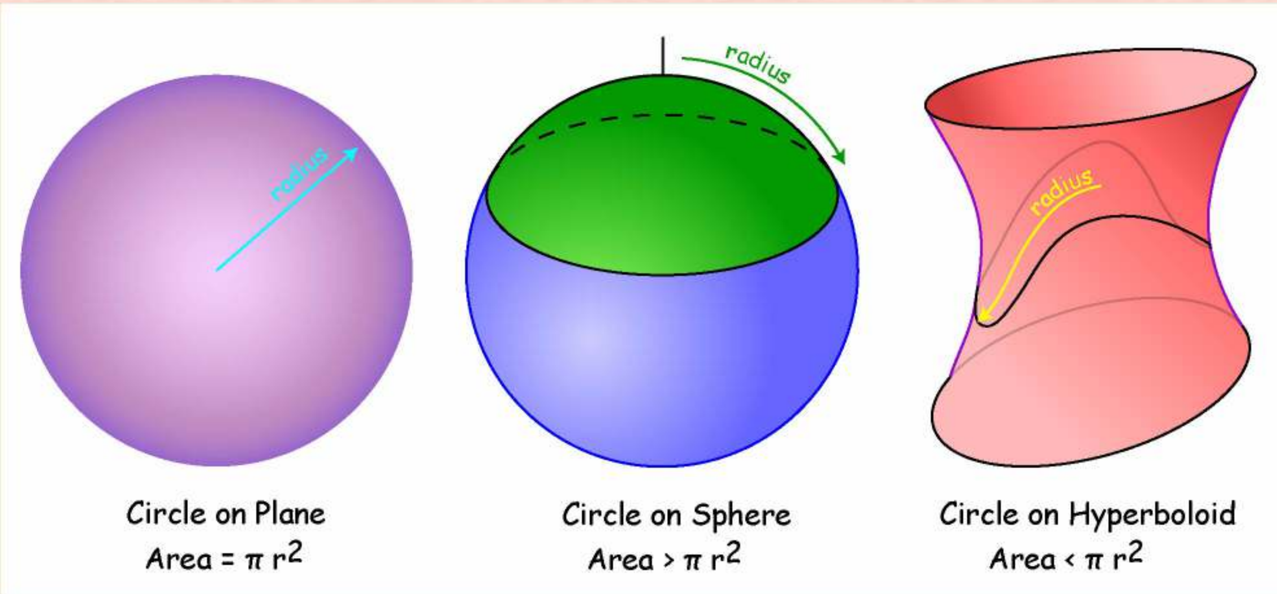
- In flat euclidean space you've gotten used to the idea that the area of a circle is πr^2 and its circumference is $2\pi r$,
  where r is the radius of the circle.



Circle on Plane
Area = πr^2

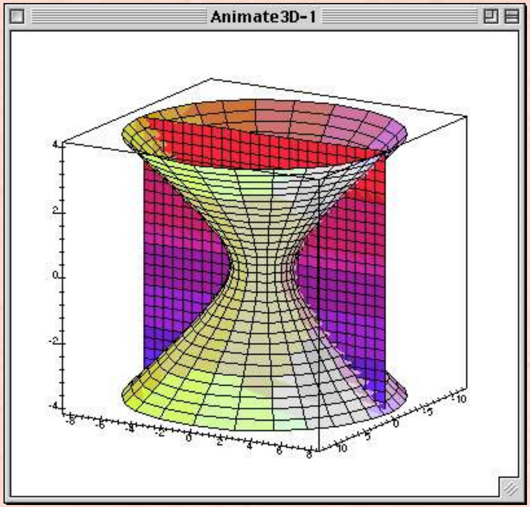
Non-Euclidean

- In non-euclidean geometry, the idea of a circle still makes sense, but the relationships between area and circumference with radius don't hold.



Hyperbolic space

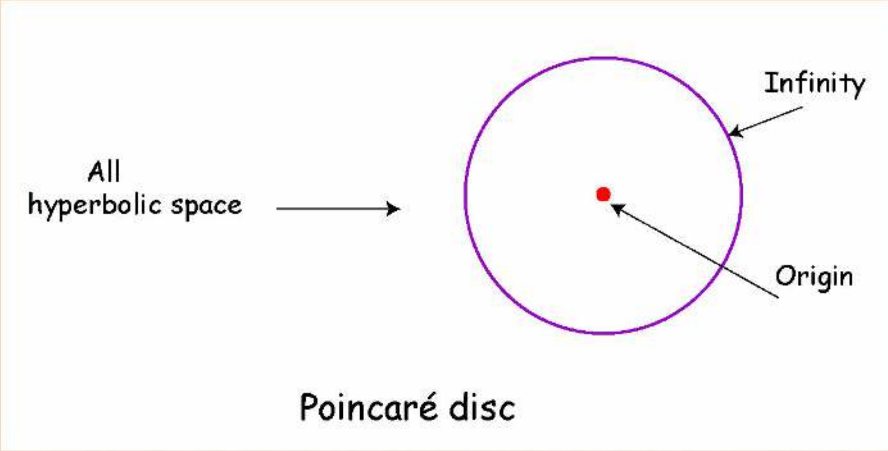
- There are a few ways to represent hyperbolic space on a flat piece of paper.



- We'll take a look at one called the Poincaré disc.

Hyperbolic space

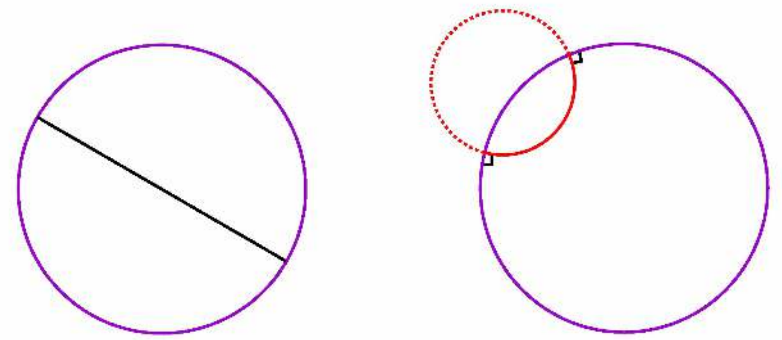
- In this model, the entire hyperbolic plane is mapped into a disk.
  Points on the boundary circle are considered as infinitely far away.



- If you imagine walking from the center of the disk (the origin) outwards, your legs get shorter and shorter, so the boundary circle is unreachable.

Hyperbolic space

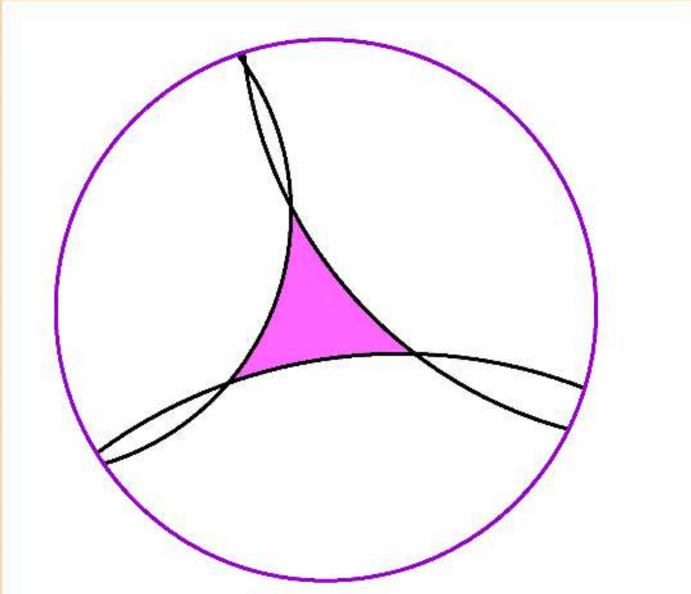
- A point in hyperbolic space is represented by a point within the disk.
- The shortest distance between two points that goes through the origin are represented by diameters of the Poincaré disk.
- The shortest distance between two points not passing through the center are arcs of circles that meet the boundary circle at right angles.



Geodesics of the Poincaré disc

Hyperbolic space

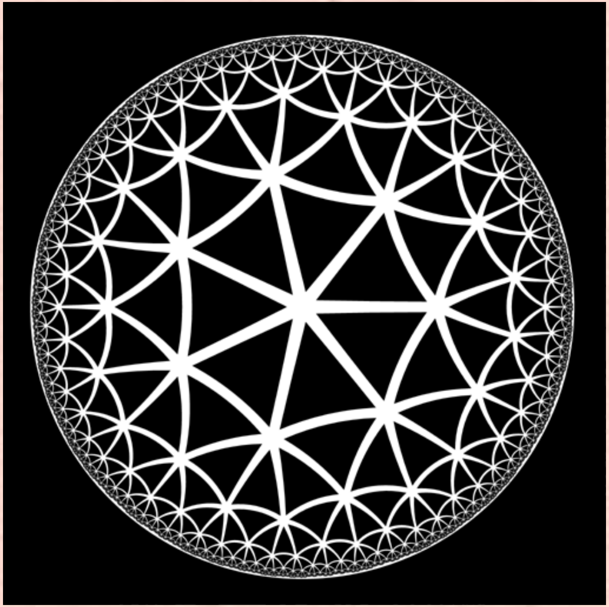
- A hyperbolic triangle is gives by the intersection of three geodesics.



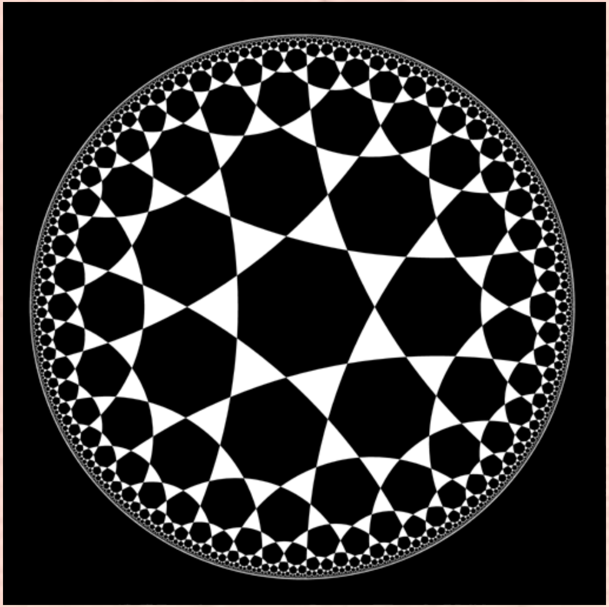
Triangles in the Poincaré disc

Hyperbolic space

- Tilings can be formed in hyperbolic space and represented in the Poincaré disc. Non-uniform distances (legs getting shorter), means the motifs are congruent.



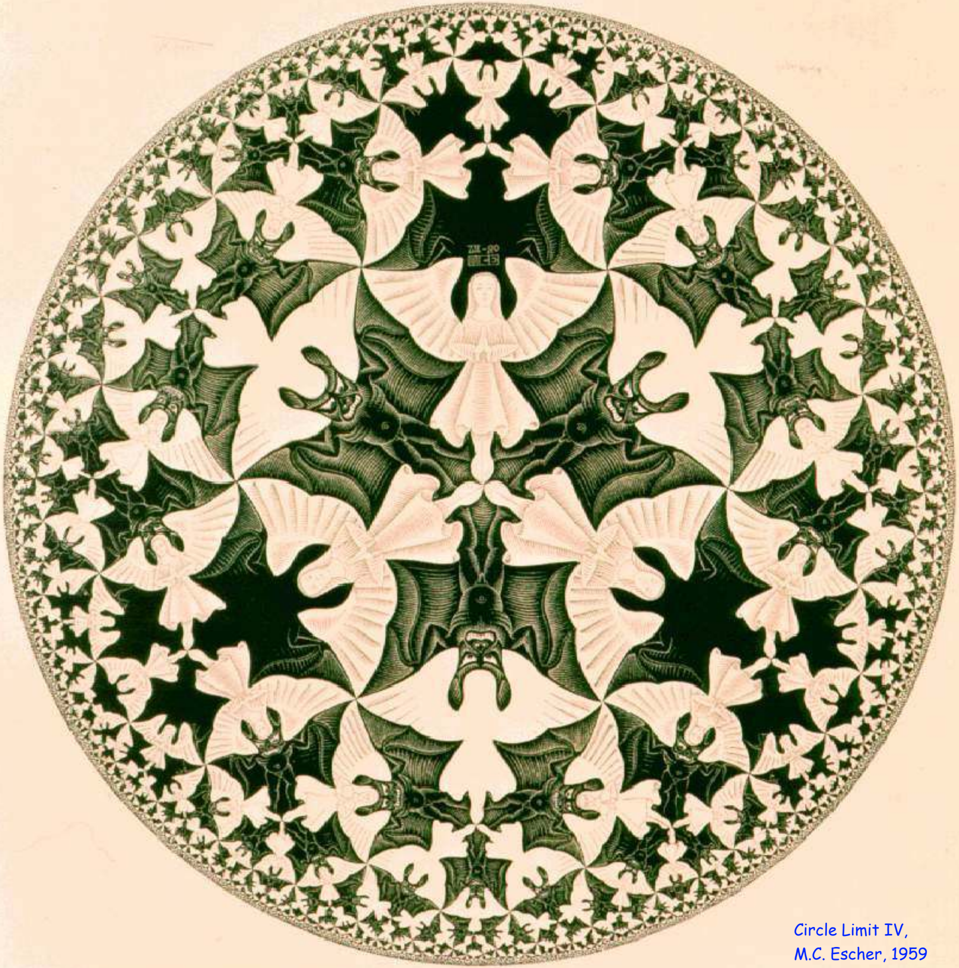
Hyperbolic tiling with 7 equilateral triangles at each vertex.



Hyperbolic equivalent of an icosadodecahedron.

- This model of hyperbolic space inspired Escher to produce a series of tesselations on woodcuts called Circle Limit.

Circle Limit III,
M.C. Escher, 1959

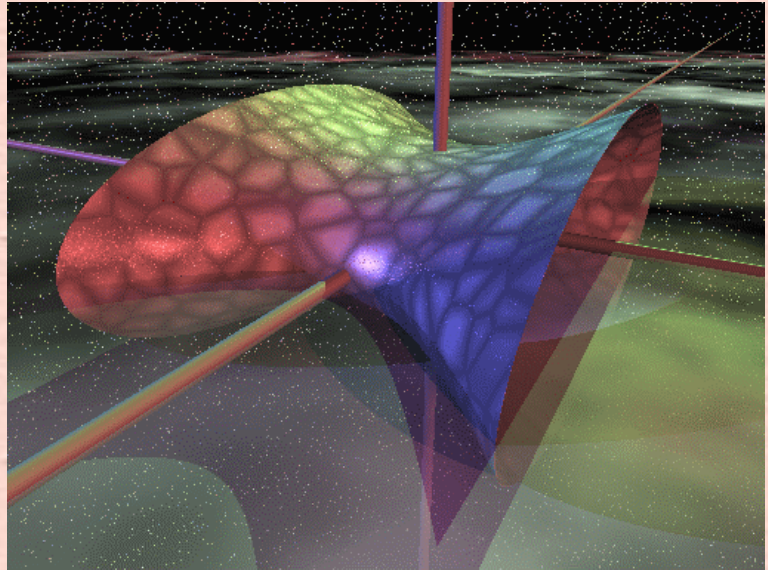


Circle Limit IV,
M.C. Escher, 1959

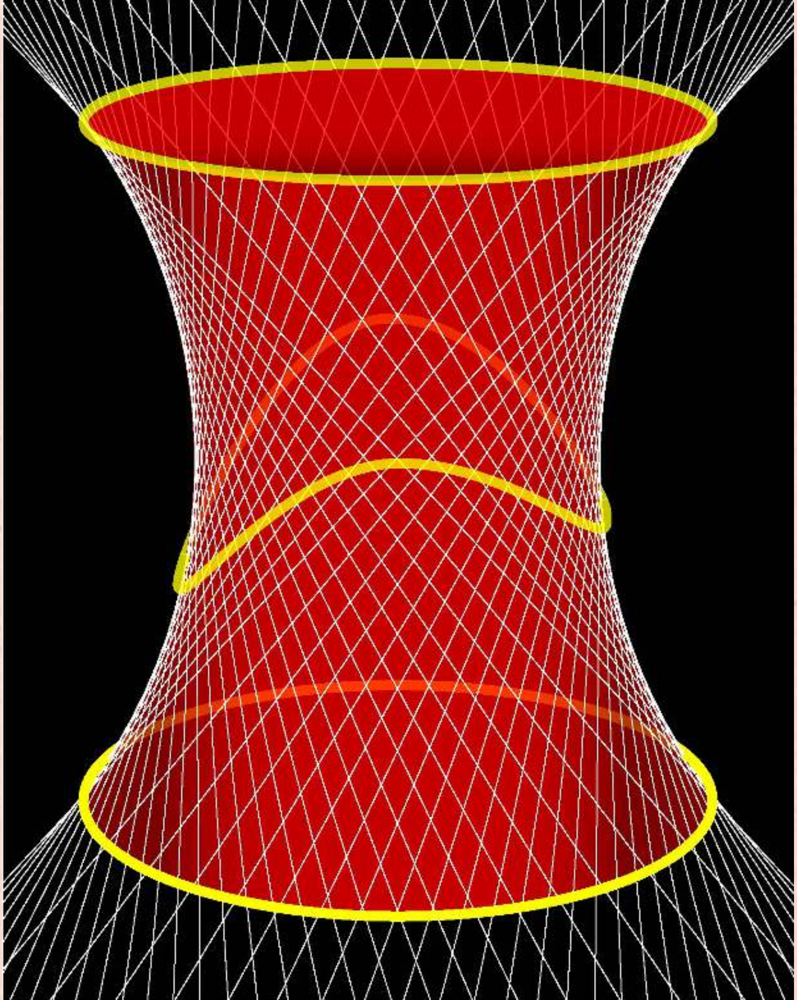
Hyperbolic space

- Adding up all the matter that we can detect, most studies suggest the geometry of the matter in the universe is hyperbolic.

- Hence, the universe expands forever, thermodynamics wins, and the final fate is ice.

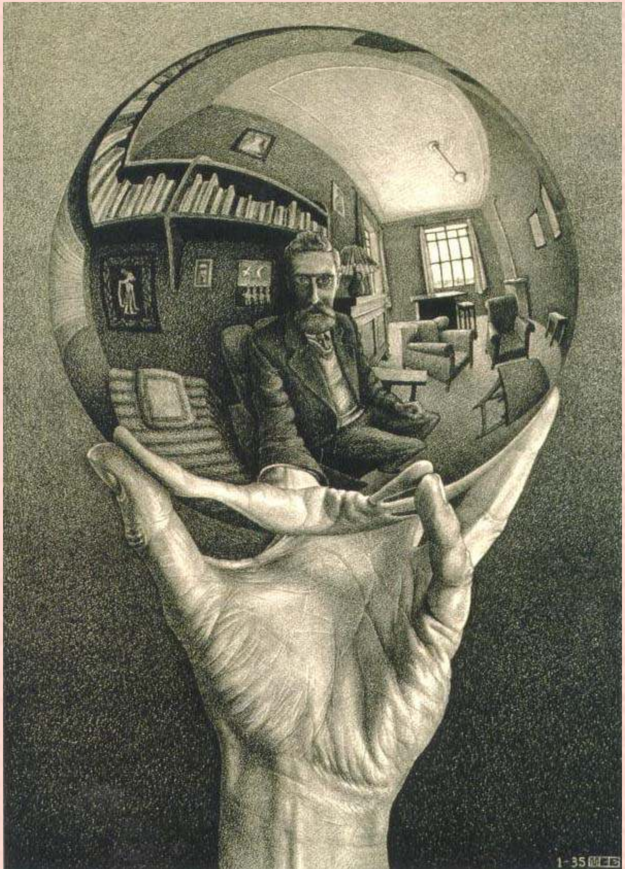


- If you want more details, take my astronomy class!



Elliptical space

- We can use the surface of a sphere as a model of elliptic geometry.



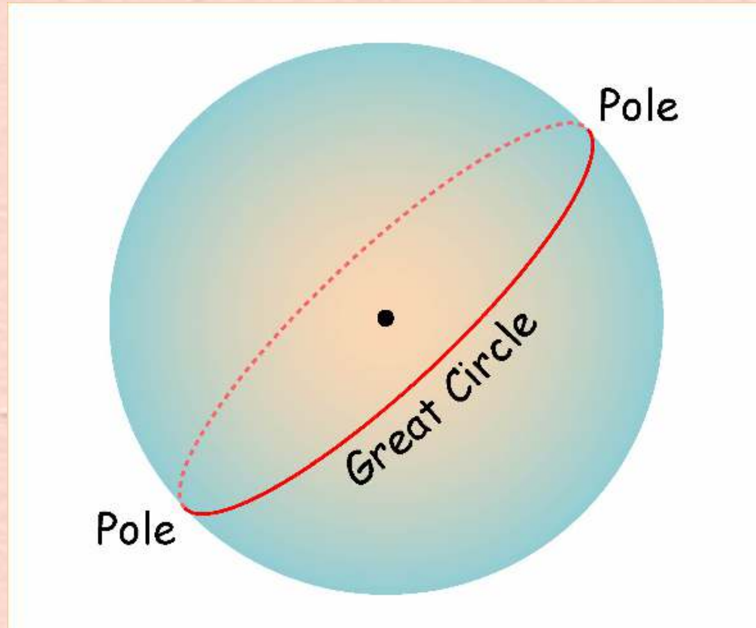
A Hand With Reflecting Sphere
M. C. Escher, Litografía, 1935.

Elliptical space

- A great circle is formed on the surface of a sphere by a plane passing through the center of the sphere.

- For example, the equator is a great circle on the surface of the Earth.

- Each pair of points on the opposite ends of a diameter are called pole points.

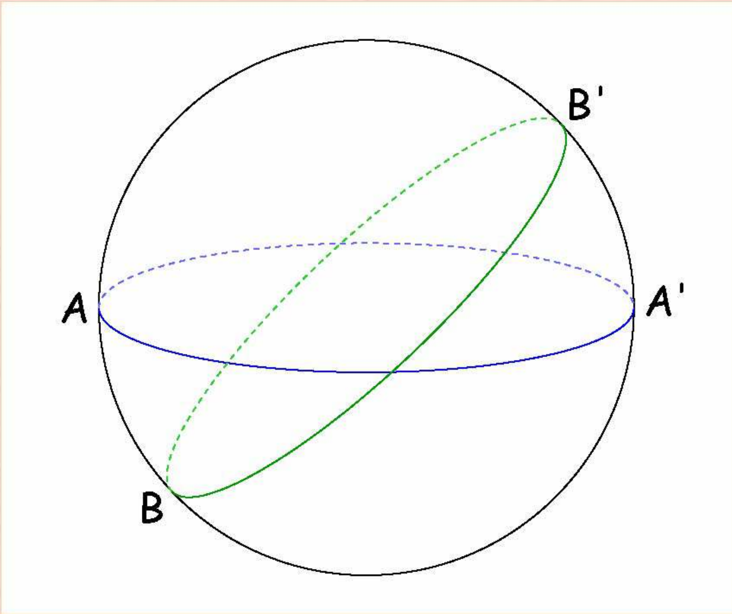


Elliptical space

- Points A and A′ are pole points.
  So are points B and B′.

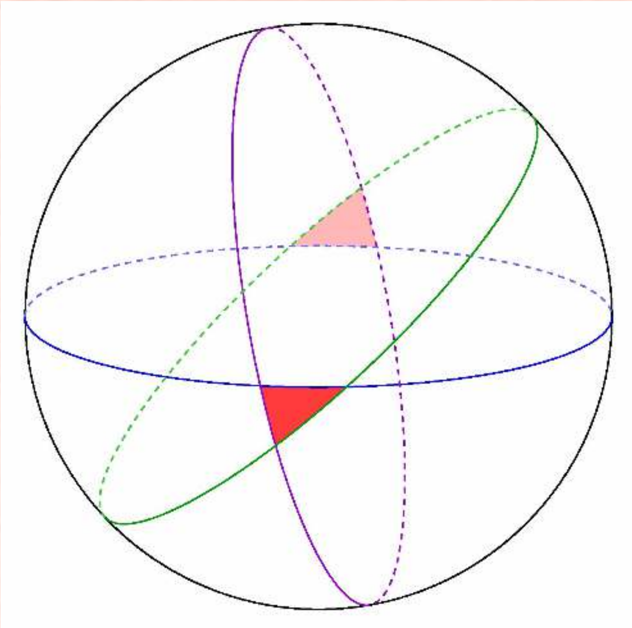
- More than one great circle can pass through points A and A′.

- However, only one great circle can pass through all four points.



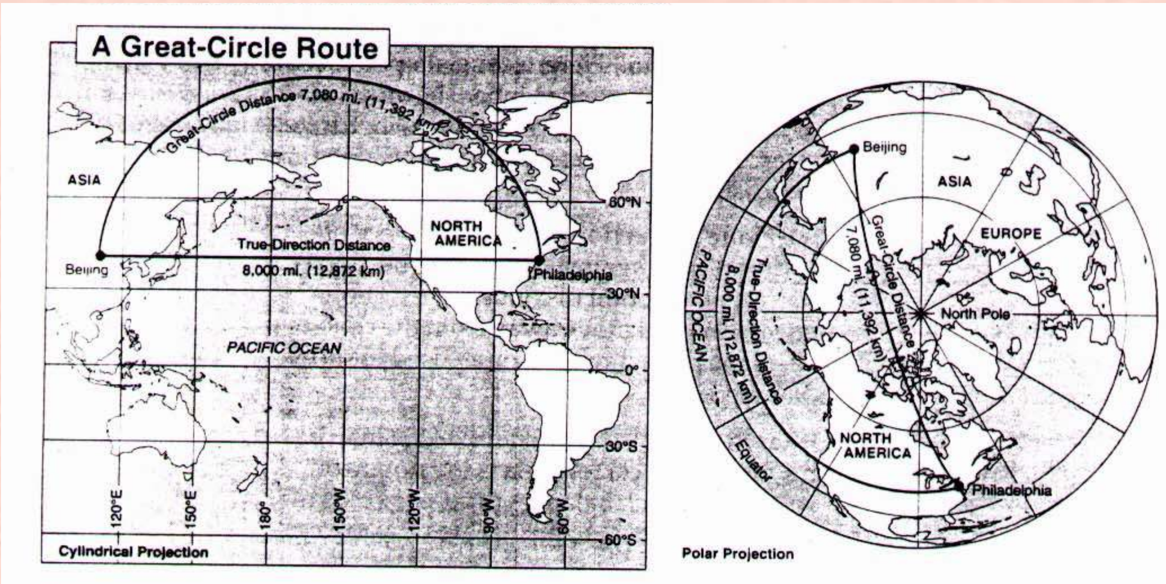
Elliptical space

- A spherical triangle is gives by the intersection of three geodesics.



Elliptical space

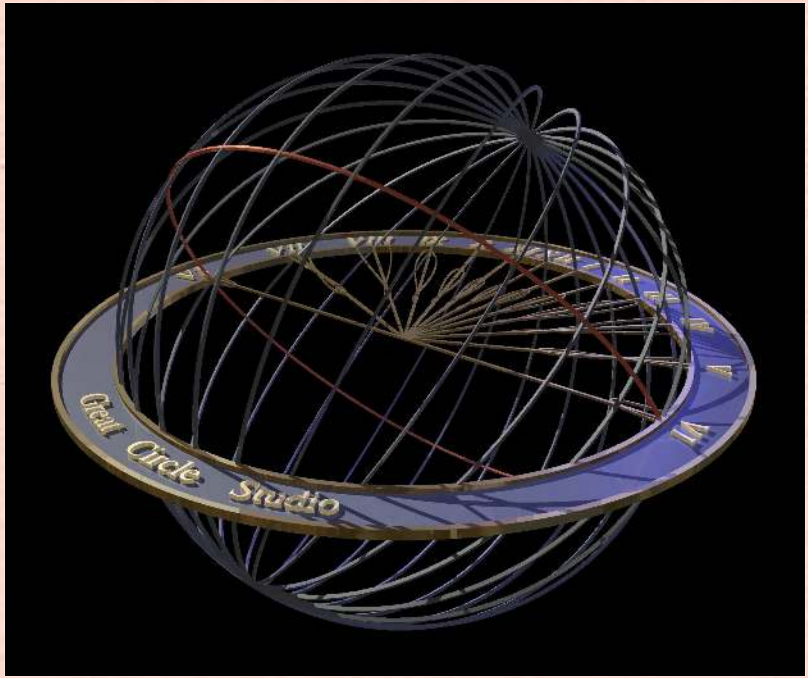
- The geodesic in a spherical space, the shortest distance between two points, is simply an arc of a great circle.



Elliptical space

- In Euclidean geometry, lines never end and are infinite in length.
  In spherical geometry, great circles never end; however their length is finite!

- If you travel the shortest path along a sphere, you will eventually return to your starting point.



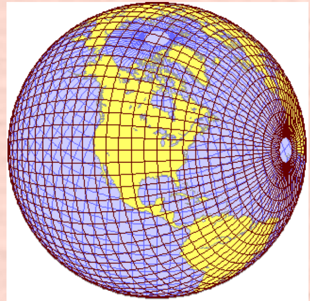
Elliptical space

- All lines that begin parallel eventually converge, so there are no parallel lines in spherical or elliptical geometries.

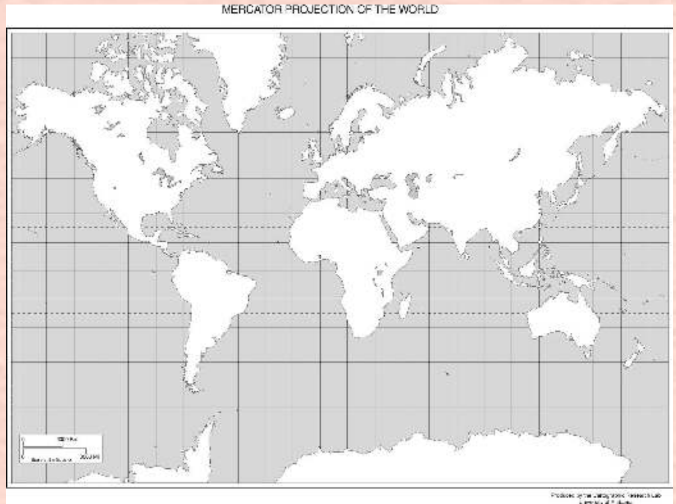
Sphere Spirals,
M.C. Escher, 1958, Woodcut

Elliptical space

- Perhaps the most important application of spherical geometry is making maps of the Earth or any other planet.

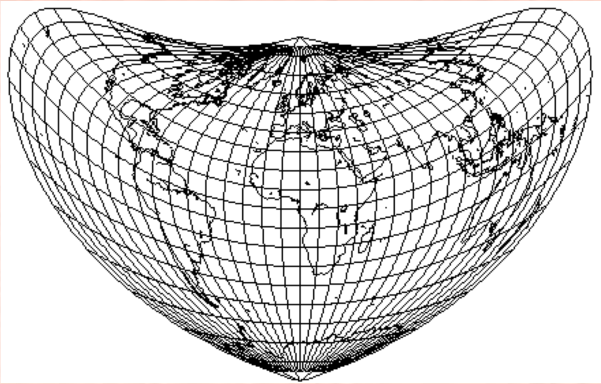


- How does one project the features on the surface of a sphere onto a flat, Euclidean piece of paper or computer screen?



Playtime

- During today's in-class construction, you'll discover some of the ways we have flattened the Earth over the last 3000 years to make maps.



Bonne projection.
Conic, equal-area. All parallels are divided truly and the connecting curves make the meridians. Scale is true along the central meridian and along all parallels.